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Differential Equations as a Projection of Implicit Functions Using Spatio-Temporal Taylor Expansion and Critical Points Properties

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Abstract. This contribution introduces a novel method for formulating differential equations. This method relies on expanding an implicit function that varies with time (denoted as "t") in the space-time domain using Taylor series. This formulation encompasses both ordinary differential equations (ODEs) and partial differential equations (PDEs).In the context of visualizing vector fields, such as fluid flow and electromagnetic fields, the critical points of ODEs play a crucial role in understanding physical phenomena behavior. This paper outlines a general approach for formulating ODEs and PDEs by treating them as time-varying scalar functions using the Taylor expansion. Furthermore, a new condition for identifying critical points is derived and specified specifically for cases where the function is invariant with respect to time (referred to as "t-invariant"). This newly derived formula enhances the detection of critical points, particularly in the context of acquiring and analyzing large 3D fluid flow data. This advancement enables efficient compression of 3D vector data and their representation through radial basis functions (RBFs).

Keywords: Numerical methods (mathematics), differential equations, critical points, implicit functions, partial differential equations, linear algebra, critical points, implicit functions, radial basis functions.
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INTRODUCTION

Most computational and physical problems commonly result in the need to formulate solutions using either partial differential equations (PDEs) or ordinary differential equations (ODEs). Various methods are employed to solve these equations. It's important to note that solving PDEs or ODEs differs significantly from solving algebraic equations. In the case of PDEs, a solution necessitates the specification of boundary conditions, which provide essential information about the behavior of the equation at the boundaries of the problem domain. On the other hand, when dealing with ODEs, initial conditions must be provided. These initial conditions offer the starting point for solving the differential equation. ODEs are typically represented as follows:

$$\frac{d\boldsymbol{x}}{dt} = \boldsymbol{f}(\boldsymbol{x}(t), t)$$

or in an implicit ordinary differential form as:

$$F\left(\boldsymbol{x}, \frac{d\boldsymbol{x}}{dt}, \cdots, \frac{d^n\boldsymbol{x}}{dt^n}, t\right) = 0$$

Typically, the function f(or F) does not depend on time; it is considered time-independent. In the context of partial differential equations (PDEs), the differential equation is often expressed in implicit form as follows:

$$F\left(\boldsymbol{x}, u, \frac{\partial u}{\partial x_i}, \frac{\partial^2 u}{\partial x_i \partial x_j}, \frac{\partial^3 u}{\partial x_i \partial x_j \partial x_k}, \dots, t\right) = 0$$

Let's delve into both cases in more detail, and for clarity, we'll use the following notation:

a – scalar value $\quad$ $\boldsymbol{a}$ vector $\quad$ $\boldsymbol{A}$ matrix $\quad$ $\boldsymbol{\xi} = (\boldsymbol{x}, t)$ $\quad$ $\boldsymbol{\eta} = (\boldsymbol{x}_0, t_0)$

In both cases, namely ordinary differential equations (ODEs) and initial conditions, they can be formulated as follows:

$$F(\boldsymbol{x}(t), t) = 0 \qquad \dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}(t), t) \qquad \boldsymbol{x}_0 = \boldsymbol{x}(0)$$

Now, the function $F(\boldsymbol{x}(t), t)$ has the following derivatives (assuming, that $F_{\boldsymbol{x}t} = F_{t\boldsymbol{x}}$.):

$$\frac{\partial F(\boldsymbol{x}(t), t)}{\partial \boldsymbol{x}} = F_x = \boldsymbol{g}(\boldsymbol{x}, t) \qquad \frac{\partial F(\boldsymbol{x}(t), t)}{\partial t} = F_t = h(\boldsymbol{x}, t) \qquad \frac{\partial (\boldsymbol{g}(\boldsymbol{x}(t), t)}{\partial t} = \boldsymbol{g}_t$$

$$\frac{\partial^2 F(\boldsymbol{x}(t), t)}{\partial \boldsymbol{x}^2} = F_{\boldsymbol{x}\boldsymbol{x}} = \boldsymbol{G}_x(\boldsymbol{x}, t) \qquad \frac{\partial^2 F(\boldsymbol{x}(t), t)}{\partial t^2} = F_{tt} = h_t(\boldsymbol{x}, t) \qquad \frac{\partial (h(\boldsymbol{x}(t), t)}{\partial \boldsymbol{x}} = \boldsymbol{h}_{\boldsymbol{x}}$$

Let us explore time dependency

$$\frac{d}{dt}\left[\frac{\partial F(\boldsymbol{x},t)}{\partial \boldsymbol{x}}\right] = \frac{d}{dt}[\boldsymbol{g}(\boldsymbol{x},t)] = \frac{\partial \boldsymbol{g}(\boldsymbol{x},t)}{\partial \boldsymbol{x}}\frac{d\boldsymbol{x}}{dt} + \frac{\partial \boldsymbol{g}(\boldsymbol{x},t)}{\partial t} = \boldsymbol{G}_x\dot{\boldsymbol{x}} + \boldsymbol{g}_t$$

$$\frac{d}{dt}\left[\frac{\partial F(\boldsymbol{x},t)}{\partial t}\right] = \frac{d}{dt}[h(\boldsymbol{x},t)] = \frac{\partial h(\boldsymbol{x},t)}{\partial \boldsymbol{x}}\frac{d\boldsymbol{x}}{dt} + \frac{\partial h(\boldsymbol{x},t)}{\partial t} = \boldsymbol{h}_x\dot{\boldsymbol{x}} + h_t$$

When we perform a Taylor expansion in time for functions that vary with respect to time (t), we obtain the following:

$$F(\boldsymbol{\xi}) = F(\boldsymbol{\eta}) + \frac{dF(\boldsymbol{\eta})}{dt}(t-t_0) + \frac{1}{2}\frac{d^2F(\boldsymbol{\eta})}{dt^2}(t-t_0)^2 + \cdots ..$$

Just as the value of t changes over time, the function $\boldsymbol{x}(t)$ also changes. Consequently, the first derivative with respect to time can be expressed as:

$$\frac{dF(\boldsymbol{\eta})}{dt} = \frac{\partial F(\boldsymbol{x},t)}{\partial \boldsymbol{x}}\frac{d\boldsymbol{x}}{dt} + \frac{\partial F(\boldsymbol{x},t)}{\partial t}$$

and the second derivative as:

$$\begin{aligned}\frac{d^2F(\boldsymbol{\eta})}{dt^2} &= \frac{d}{dt}\left[\frac{dF(\boldsymbol{\eta})}{dt}\right] = \frac{d}{dt}\left[\frac{\partial F(\boldsymbol{x},t)}{\partial \boldsymbol{x}}\frac{d\boldsymbol{x}}{dt} + \frac{\partial F(\boldsymbol{x},t)}{\partial t}\right] = \frac{d}{dt}\left[\boldsymbol{g}(\boldsymbol{x},t)\frac{d\boldsymbol{x}}{dt} + h(\boldsymbol{x},t)\right] \\ &= \frac{\partial \boldsymbol{g}(\boldsymbol{x},t)}{\partial \boldsymbol{x}}\frac{d\boldsymbol{x}}{dt}\cdot\frac{d\boldsymbol{x}}{dt} + \boldsymbol{g}(\boldsymbol{x},t)\frac{d^2\boldsymbol{x}}{dt^2} + \frac{dh(\boldsymbol{x},t)}{dt} \\ &= \frac{\partial \boldsymbol{g}(\boldsymbol{x},t)}{\partial \boldsymbol{x}}\frac{d\boldsymbol{x}}{dt}\cdot\frac{d\boldsymbol{x}}{dt} + \boldsymbol{g}(\boldsymbol{x},t)\frac{d^2\boldsymbol{x}}{dt^2} + \frac{\partial h(\boldsymbol{x},t)}{\partial \boldsymbol{x}}\frac{d\boldsymbol{x}}{dt} + \frac{\partial h(\boldsymbol{x},t)}{\partial t} \\ &= \boldsymbol{G}_x(\boldsymbol{x},t)\dot{\boldsymbol{x}}^2 + \boldsymbol{g}(\boldsymbol{x},t)\ddot{\boldsymbol{x}} + \boldsymbol{h}_x(\boldsymbol{x},t)\dot{\boldsymbol{x}} + h_t(\boldsymbol{x},t) \\ &= F_{\boldsymbol{xx}}(\boldsymbol{x},t)\dot{\boldsymbol{x}}^2 + F_x(\boldsymbol{x},t)\ddot{\boldsymbol{x}} + F_{\boldsymbol{x}t}(\boldsymbol{x},t)\dot{\boldsymbol{x}} + F_{tt}(\boldsymbol{x},t)\end{aligned}$$

Using the Taylor expansion for the function $F(\boldsymbol{x}(t),t)$:

$$F(\boldsymbol{x}(t),t) = F(\boldsymbol{\eta}) + \frac{dF(\boldsymbol{\eta})}{dt}(t-t_0) + \frac{1}{2}\frac{d^2F(\boldsymbol{\eta})}{dt^2}(t-t_0)^2 + R_n = 0$$

We can observe that the following identity holds true by definition:

$F(\boldsymbol{\xi}) = F(\boldsymbol{x}(t),t) = 0 \quad \forall t \geq 0$ i.e. $F(\boldsymbol{\xi}) = F(\boldsymbol{\eta}) = 0$

By applying the Taylor expansion and the aforementioned identities, we can derive the following expressions using both linear and quadratic elements:

$$\frac{dF(\boldsymbol{\eta})}{dt}(t-t_0) + \frac{1}{2}\frac{d^2F(\boldsymbol{\eta})}{dt^2}(t-t_0)^2 + R_n = 0$$

Rewriting that the following is obtained:

$$\left(\frac{\partial F(\boldsymbol{x},t)}{\partial \boldsymbol{x}}\frac{d\boldsymbol{x}}{dt} + \frac{\partial F(\boldsymbol{x},t)}{\partial t}\right)(t-t_0) + \frac{1}{2}\left(\boldsymbol{G}_x(\boldsymbol{x},t)\dot{\boldsymbol{x}}^2 + \boldsymbol{g}(\boldsymbol{x},t)\ddot{\boldsymbol{x}} + \boldsymbol{h}_x(\boldsymbol{x},t)\dot{\boldsymbol{x}} + h_t(\boldsymbol{x},t)\right)(t-t_0)^2 + R_n = 0$$

It should be noted, that $\boldsymbol{A}\boldsymbol{x}^2$ should be read as a quadratic form, i.e. $\boldsymbol{x}^T\boldsymbol{A}\boldsymbol{x}$. Then

$$(\boldsymbol{g}(\boldsymbol{x},t)\dot{\boldsymbol{x}} + h(\boldsymbol{x},t) + \boldsymbol{G}_x(\boldsymbol{x},t)\dot{\boldsymbol{x}}^2)(t-t_0) + \frac{1}{2}\left(\boldsymbol{g}(\boldsymbol{x},t)\ddot{\boldsymbol{x}} + \boldsymbol{h}_x(\boldsymbol{x},t)\dot{\boldsymbol{x}} + h_t(\boldsymbol{x},t)\right)(t-t_0)^2 + R_n = 0$$

When analyzing the behavior of 2D and 3D physical phenomena, such as fluid flow and electromagnetic fields, it is essential to examine critical points within the relevant ordinary differential equations (ODEs). These critical points are characterized by $\dot{\boldsymbol{x}} = \boldsymbol{0}$, indicating that the derivative of x with respect to time is zero. By employing the linear and quadratic elements of the Taylor expansion, we arrive at the following equations to describe these critical points:

$$h(\boldsymbol{x},t) + \frac{1}{2}\left(\boldsymbol{g}(\boldsymbol{x},t)\ddot{\boldsymbol{x}} + h_t(\boldsymbol{x},t)\right)(t-t_0) = 0$$

In the case of t-invariant systems, i.e. $F\big(\boldsymbol{x}(t)\big) = 0$ and a pro $t \neq t_0$, the following is obtained:

$$\boldsymbol{g}(\boldsymbol{x},t)\ddot{\boldsymbol{x}} = 0 \qquad\qquad \frac{\partial F(\boldsymbol{x})}{\partial \boldsymbol{x}}\ddot{\boldsymbol{x}} = 0$$

It leads to a new condition:

$$F_x(\boldsymbol{x}(t))\ddot{\boldsymbol{x}} = 0$$

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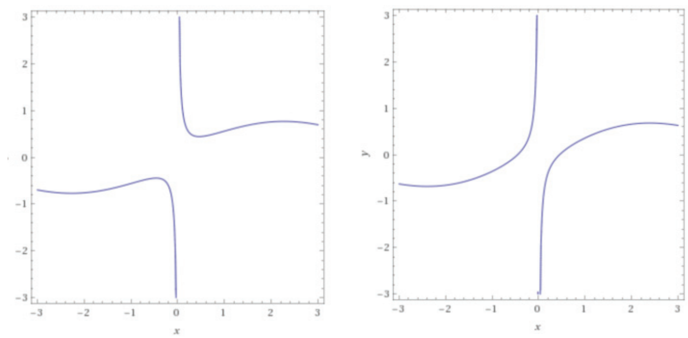
This result holds great significance as it enhances the accuracy and reliability of detecting critical points, a crucial requirement for interpolating and approximating large and intricate 2D and 3D vector fields. This advancement is particularly valuable in applications discussed by Smolik[8][9] and Skala[6].

EXAMPLE

Let us consider a differential equation

$x\dot{y} + y = \sin x$ and its solution $y = \frac{1}{x}(c - \cos x)$ $x \neq 0$



$c = 1.1$ $c = 0.9$

Fig.1 Function $F(\boldsymbol{x}) = 0$ behavior for different values c

In this case the implicit function

$$F(\boldsymbol{x}) = F(x, y) = (c - \cos x) - xy = 0$$

the derivatives are

$$\frac{\partial F(\boldsymbol{x})}{\partial \boldsymbol{x}} = F_{\boldsymbol{x}} = [\sin x - y, -x] \qquad \frac{\partial F(\boldsymbol{x})}{\partial \boldsymbol{x}}\ddot{\boldsymbol{x}} = (\sin x - y)\ddot{x} - x\ddot{y} = 0$$

Using a trick $x_1 = x$, $x_2 = y$, the differential equation can be rewritten as:

$$\dot{x}_1 = 1 \qquad \dot{x}_2 = (\sin x_1 - x_2)/x_1$$

Verification

$$\dot{x}_1 = 1 \qquad \ddot{x}_1 = 0 \qquad x_1 \in R^1 \;\&\; x_1 \neq 0 \quad x_2 \in R^1$$

$$\dot{x}_2 = (\sin x_1 - x_2)/\,x_1 \qquad \ddot{x}_2 = [\dot{x}_1(x_2 - \sin \dot{x}_1 + x_1 \cos x_1) - x_1\dot{x}_2]/x_1^2$$

Applying the condition for t-invariant ordinary differential equations (ODEs), we can now proceed to:

$$\frac{\partial F(\boldsymbol{x})}{\partial \boldsymbol{x}}\ddot{\boldsymbol{x}} = (\sin x - y)\ddot{x} - x\ddot{y} = 0$$

$$\frac{\partial F(\boldsymbol{x})}{\partial \boldsymbol{x}}\ddot{\boldsymbol{x}} = \frac{x_1[\dot{x}_1(x_2 - \sin \dot{x}_1 + x_1 \cos x_1) - x_1\dot{x}_2]}{x_1^2} = \frac{\dot{x}_1(x_2 - \sin \dot{x}_1 + x_1 \cos x_1) - x_1\dot{x}_2}{x_1}$$

$$= \frac{\dot{x}_1(x_2 - \sin \dot{x}_1)}{x_1} + \cos x_1 - \dot{x}_2 = 0$$

As $\dot{x}_1 = 1$, then

$$\frac{\partial F(\boldsymbol{x})}{\partial \boldsymbol{x}}\ddot{\boldsymbol{x}} = x_2 + \cos(1) - \sin(1) - \dot{x}_2 = 0 \qquad \frac{\partial F(\boldsymbol{x})}{\partial \boldsymbol{x}}\ddot{\boldsymbol{x}} = x_2 - (\sin(1) - \cos(1)) - \dot{x}_2 = 0$$

Using identity

$$\sin \alpha \pm \cos \beta = 2 \sin\left(\frac{\alpha \pm \beta}{2}\right) \cos\left(\frac{\alpha \mp \beta}{2}\right) \qquad \frac{\partial F(\boldsymbol{x})}{\partial \boldsymbol{x}}\ddot{\boldsymbol{x}} = x_2 - 2\sin(1).\cos(0) - \dot{x}_2 = 0$$

Since $\dot{x}_2 = 0$ is equal to zero for all critical points, we can derive the new condition for critical points of t -invariant ordinary differential equations as follows:

$$\frac{\partial F(\boldsymbol{x})}{\partial \boldsymbol{x}}\ddot{\boldsymbol{x}} = 0$$

It can be seen, that there is a connection to the Frenet-Serret formula[15].

CONCLUSION

This contribution provides a concise overview of a novel condition for identifying critical points in differential equations, employing a spatio-temporal Taylor expansion. We establish a general framework for differential equations that vary with time (t) and present specific criteria for those that are time-invariant (t-invariant). An illustrative example enclosed within this text serves as a demonstration of the proposed methodology.

In the future, we plan to apply this approach to more intricate scenarios where physical phenomena are characterized by differential equations, as discussed in works such as Skala[6], Smolik[8], etc. This will involve employing the Taylor expansion for vector data, as outlined in Skala[16], without the need for tensor representation. Also, use of meshless methods for ODE and PDE is a challenging topic, Skala[18].

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