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Critical Points Properties of Ordinary Differential Equations as a Projection of Implicit Functions Using Spatio-Temporal Taylor Expansion *

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Abstract. This contribution describes a new approach to formulation of ODE and PDE critical points using implicit formulation as t-variant scalar function using the Taylor expansion. A general condition for the critical points is derived and specified for t invariant case. It is expected, that the given new formulae lead to more reliable detection of critical points especially for large 3D fluid flow data acquisition, which enable high $3D$ vector compression and their representation using radial basis functions (RBF).
In the case of vector field visualization, e.g. fluid flow, electromagnetic fields, etc., the critical points of ODE are critical for physical phenomena behavior.

Keywords: Critical points · Vector fields visualization · Numerical methods · Ordinary differential equations · Partial differential equations · Implicit functions · Radial basis functions

1 Introduction

Many physical and computational problems lead to formulations using ordinary differential equations (ODE) or partial differential equations(PDE) and different methods are used for their solution. Methods for solution of ODEs or PDEs are quite different from a solution of algebraic equations, as the solution requires specification of the initial conditions in the case of ODE, while in the case of PDE the boundary conditions have to be specified.

In the case of vector field visualization, e.g. fluid flow, electromagnetic fields, etc., the critical points of ODE are critical for physical phenomena behavior. The critical or null points are points in which the derivative $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}(t))$ is $zero$, see Lebl[12], Smolik[16] and Skala[15]. Classification of such points helps in vector

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fields visualization Helman[7], Koch[11], Schuermann[13], Skala[15], Huang[8] and Smolik[16].

In the ODE case differential equations are usually given as:

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}(t), t) \tag{1}$$

or in a more general implicit ordinary differential form as:

$$F(\mathbf{x}, \frac{d\mathbf{x}}{dt}, \cdots, \frac{d^n\mathbf{x}}{dt^n}, t) = 0 \tag{2}$$

with initial conditions $\mathbf{x}_0 = [x_0(0), \ldots, x_n(0)]^T$. The implicit forms from the geometrical point of view was studied in Goldman[5][6] and Agoston[1]. In the majority of cases, the function f, resp. F are time independent, i.e. $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}(t))$.

In the case of the partial differential equations (PDE), the differential equation is given in the implicit form as:

$$F(\mathbf{x}, u, \frac{\partial u}{\partial x_i}, \frac{\partial^2 u}{\partial x_i \partial x_j}, \frac{\partial^3 u}{\partial x_i \partial x_j \partial x_k}, \ldots, \frac{\partial u}{\partial t}, \frac{\partial^2 u}{\partial t^2}, \ldots, t) = 0 \tag{3}$$

where $\mathbf{x} = [x_1, \ldots, x_n]^T$ [1] are points fixed position in the domain, derivatives are in the domain axes direction, i.e. in the 3D case:

$$\frac{\partial u}{\partial x_1} \equiv \frac{\partial u}{\partial x} \qquad \frac{\partial u}{\partial x_2} \equiv \frac{\partial u}{\partial y} \qquad \frac{\partial u}{\partial x_3} \equiv \frac{\partial u}{\partial z} \tag{4}$$

Let us explore the ordinary differential equation case more in a detail. The following notation will be used:

- a - scalar value,
- $\mathbf{a}$ - vector,
- $\mathbf{A}$ - matrix,
- $\xi = (\mathbf{x}, t)$ spatio-temporal vector, i.e. $\mathbf{x} = [x_1, \ldots, x_n]^T$, $\xi = [x_1, \ldots, x_n, t]^T$
- $\eta = (\mathbf{x}_0, t_0)$

It should be noted, that $\mathbf{x}_i$ and t have different physical units, i.e. $[m]$ and $[s]$. In the following the ordinary differential equations (ODE) are explored.

2 Ordinary differential equations and Taylor expansion

The ordinary differential equation (ODE) can be formulated as:

$$F(\mathbf{x}(t), t) = 0 \quad \text{or} \quad \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}(t), t) \quad \text{with} \quad \mathbf{x}_0 = \mathbf{x}(0) \tag{5}$$

where $\mathbf{x}_0 = [x_1(0), \ldots, x_n(0)]^T$ and n is the dimensionality of the ODE.

[1] It should be noted that in the case of the PDE the coordinates $\mathbf{x}$ are assumed to be time independent

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Now, the function $F(\mathbf{x}(t), t)$ has the following derivatives:

$$\frac{\partial F(\mathbf{x}(t),t)}{\partial \mathbf{x}} = F_{\mathbf{x}} = \mathbf{g}(\mathbf{x},t) \qquad \frac{\partial F(\mathbf{x}(t),t)}{\partial t} = F_t = h(\mathbf{x},t) \tag{6}$$

$$\frac{\partial \mathbf{g}(\mathbf{x}(t),t)}{\partial t} = \mathbf{g}_t \qquad \frac{\partial^2 F(\mathbf{x}(t),t)}{\partial \mathbf{x}^2} = F_{\mathbf{x}\mathbf{x}} = \mathbf{G}_x(\mathbf{x},t) \tag{7}$$

$$\frac{\partial^2 F(\mathbf{x}(t),t)}{\partial t^2} = F_{tt} = h_t(\mathbf{x},t) \qquad \frac{\partial (h(\mathbf{x}(t),t)}{\partial \mathbf{x}} = \mathbf{h}_{\mathbf{x}} \tag{8}$$

assuming, that $F_{\mathbf{x}t} = F_{t\mathbf{x}}$.

Let us explore the time dependency;

$$\begin{aligned} \frac{d}{dt}\left[\frac{\partial F(\mathbf{x},t)}{\partial \mathbf{x}}\right] &= \frac{d}{dt}[\mathbf{g}(\mathbf{x},t)] = \frac{\partial \mathbf{g}(\mathbf{x},t)}{\partial \mathbf{x}}\frac{d\mathbf{x}}{dt} + \frac{\partial \mathbf{g}(\mathbf{x},t)}{\partial t} \\ &= \mathbf{G}_{\mathbf{x}}\dot{\mathbf{x}} + \mathbf{g}_t \end{aligned} \tag{9}$$

$$\begin{aligned} \frac{d}{dt}\left[\frac{\partial F(\mathbf{x},t)}{\partial t}\right] &= \frac{d}{dt}[h(\mathbf{x},t)] = \frac{\partial h(\mathbf{x},t)}{\partial \mathbf{x}}\frac{d\mathbf{x}}{dt} + \frac{\partial h(\mathbf{x},t)}{\partial t} \\ &= \mathbf{h}_{\mathbf{x}}\dot{\mathbf{x}} + h_t \end{aligned} \tag{10}$$

Then the Taylor expansion in time for t-varying functions, see Bronson[3] and Skala[14], the following is obtained:

$$F(\xi) = F(\eta) + \frac{dF(\eta)}{dt}(t - t_0) + \frac{1}{2}\frac{d^2F(\eta)}{dt^2}(t - t_0)^2 + \dots. \tag{11}$$

As in the case of change of t, also $\mathbf{x}(t)$ is changed. Therefore, the first derivative in time is expressed as:

$$\frac{dF(\eta)}{dt} = \frac{\partial F(\mathbf{x},t)}{\partial \mathbf{x}}\frac{d\mathbf{x}}{dt} + \frac{\partial F(\mathbf{x},t)}{\partial t} \tag{12}$$

and the second derivative as:

$$\begin{aligned} \frac{d^2F(\eta)}{dt^2} &= \frac{d}{dt}\left[\frac{dF(\eta)}{dt}\right] = \frac{d}{dt}\left[\frac{\partial F(\mathbf{x},t)}{\partial \mathbf{x}}\frac{d\mathbf{x}}{dt} + \frac{\partial F(\mathbf{x},t)}{\partial t}\right] \\ &= \frac{d}{dt}[\mathbf{g}(\mathbf{x},t)\frac{d\mathbf{x}}{dt} + h(\mathbf{x},t)] \\ &= \frac{\partial \mathbf{g}(\mathbf{x},t)}{\partial \mathbf{x}}\frac{d\mathbf{x}}{dt} \bullet \frac{d\mathbf{x}}{dt} + \mathbf{g}(\mathbf{x},t)\frac{d^2\mathbf{x}}{dt^2} + \frac{dh(\mathbf{x},t)}{dt} \\ &= \mathbf{G}_{\mathbf{x}}(\mathbf{x},t)\dot{\mathbf{x}}^2 + \mathbf{g}(\mathbf{x},t)\ddot{\mathbf{x}} + \mathbf{h}_{\mathbf{x}}(\mathbf{x},t)\dot{\mathbf{x}} + h_t(\mathbf{x},t) \\ &= F_{\mathbf{x}\mathbf{x}}(\mathbf{x},t)\dot{\mathbf{x}}^2 + F_{\mathbf{x}}(\mathbf{x},t)\ddot{\mathbf{x}} + F_{\mathbf{x}t}(\mathbf{x},t)\dot{\mathbf{x}} + F_{tt}(\mathbf{x},t) \end{aligned} \tag{13}$$

It should be noted, that $\mathbf{A}\mathbf{x}^2$ should be read as a quadratic form, i.e. $\mathbf{x}^T\mathbf{A}\mathbf{x}$.

Using the Taylor expansion for the function $F(\mathbf{x}(t), t)$:

$$F(\mathbf{x}(t),t) = F(\eta) + \frac{dF(\eta)}{dt}(t - t_0) + \frac{1}{2}\frac{d^2F(\eta)}{dt^2}(t - t_0)^2 + R_n = 0 \tag{14}$$

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3 Critical points

It can be seen that the following identity for the critical points is valid by definition Lebl[12]:

$$F(\xi) = F(\mathbf{x}(t), t) = 0 \quad \forall t \geq 0 \quad , \; i.e. \quad F(\xi) = F(\eta) = 0 \tag{15}$$

Using the Taylor expansion and the identities above, the following is obtained using linear and quadratic elements:

$$\frac{dF(\eta)}{dt}(t - t_0) + \frac{1}{2}\frac{d^2F(\eta)}{dt^2}(t - t_0)^2 + R_n = 0 \tag{16}$$

Rewriting that the following is obtained:

$$\begin{aligned}&\left(\frac{\partial F(\mathbf{x}, t)}{\partial \mathbf{x}}\frac{d\mathbf{x}}{dt} + \frac{\partial F(\mathbf{x}, t)}{\partial t}\right)(t - t_0)\\&+ \frac{1}{2}\left(\mathbf{G_x}(\mathbf{x}, t)\dot{\mathbf{x}}^2 + \mathbf{g}(\mathbf{x}, t)\ddot{\mathbf{x}} + \mathbf{h_x}(\mathbf{x}, t)\dot{\mathbf{x}} + h_t(\mathbf{x}, t)\right)(t - t_0)^2 + R_n = 0\end{aligned} \tag{17}$$

It should be noted, that $\mathbf{Ax}^2$ should be read as a quadratic form, i.e. $\mathbf{x}^T\mathbf{Ax}$. Then

$$\begin{aligned}&(\mathbf{g}(\mathbf{x}, t)\dot{\mathbf{x}} + h(\mathbf{x}, t) + \mathbf{G_x}(\mathbf{x}, t)\dot{\mathbf{x}}^2)(t - t_0)\\&+ \frac{1}{2}(\mathbf{g}(\mathbf{x}, t)\ddot{\mathbf{x}} + \mathbf{h_x}(\mathbf{x}, t)\dot{\mathbf{x}} + h_t(\mathbf{x}, t))(t - t_0)^2 + R_n = 0\end{aligned} \tag{18}$$

In the case of 2D and 3D physical phenomena behavior, e.g. fluid flow Helman[7], electromagnetic field Drake[4] etc., there are critical points of the relevant ODE, which have to be analyzed Koch[11], Schuermann[13], Skala[15], Smolik[16][17][20].

The critical points are defined as $\dot{\mathbf{x}} = \mathbf{0}$. Using the linear and quadratic elements of the Taylor expansion the following equations for critical points is obtained:

$$h(\mathbf{x}, t) + \frac{1}{2}(\mathbf{g}(\mathbf{x}, t)\ddot{\mathbf{x}} + h_t(\mathbf{x}, t))(t - t_0) = 0 \tag{19}$$

In the case of t-invariant systems, i.e. $F(\mathbf{x}(t)) = 0$ and a pro $t \neq t_0$, the following is obtained:

$$\mathbf{g}(\mathbf{x}, t)\ddot{\mathbf{x}} = 0 \quad , i.e. \quad \frac{\partial F(\mathbf{x})}{\partial \mathbf{x}}\ddot{\mathbf{x}} = 0 \tag{20}$$

It leads to a new condition:

$$F_{\mathbf{x}}(\mathbf{x}(t))\; \ddot{\mathbf{x}} = 0 \tag{21}$$

This is a significant result as it enables better and more reliable critical points detection needed for interpolation and approximation of large and complex 2D and 3D vector fields, e.g. Skala[15], Smolik[16][18][19].

Let us consider two simple ODE examples by Eq.22 to demonstrate critical points and behaviour of the vector fields, see Fig.1.

$$\begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix} = \begin{bmatrix}xy - 4\\ (x-4)(y-x)\end{bmatrix} \qquad \begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix} = \begin{bmatrix}2x + y^2 - 1\\ 6x - y^2 + 1\end{bmatrix} \tag{22}$$

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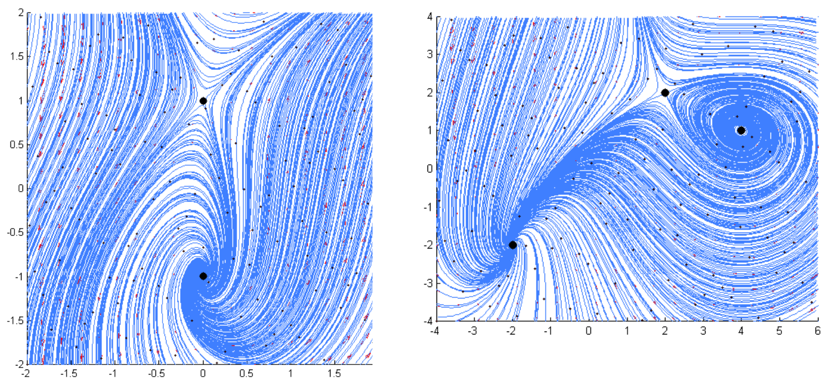


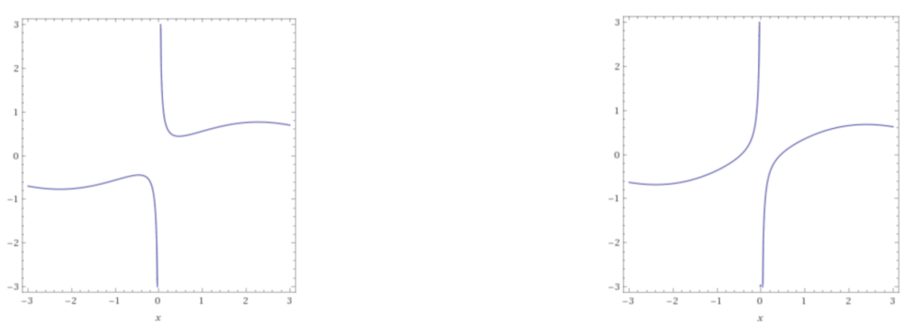
(a) Two critical points (b) Three critical points

Fig. 1: Examples of vector fields in E^2 with two and three critical points

It can be seen, that critical points have a significant influence to the vector field behaviour. It should be noted, that if a vector field is given by an acquired discrete data, specific techniques are to be used for the critical points detection and the condition given by the Eq.21 helps to robustness of this.

4 Example

Let us consider a differential equation $x\dot{y} + y = \sin(x)$ and its solution $y = \frac{1}{x}(c - \cos x)$ & $x \neq 0$. In this case the implicit function



(a) $c = 1.1$ (b) $c = 0.9$

Fig. 2: Behaviour of the function $F(\mathbf{x}) = 0$ for different values c

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$$F(\mathbf{x}) = F(x, y) = (c - \cos x) - xy = 0 \tag{23}$$

as $\mathbf{x} = [x, y]^T$. Then derivatives of the implicit function $F(\mathbf{x}) = 0$ are:

$$\frac{\partial F(\mathbf{x})}{\partial \mathbf{x}} = F_{\mathbf{x}} = [\sin x - y, -x] \qquad \frac{\partial F(\mathbf{x})}{\partial \mathbf{x}}\ddot{\mathbf{x}} = (\sin x - y)\ddot{x} - x\ddot{y} = 0 \tag{24}$$

Using a trick $x_1 = x$, $x_2 = y$, the differential equation can be rewritten as:

$$\dot{x}_1 = 1 \qquad \dot{x}_2 = (\sin x_1 - x_2)/x_1 \tag{25}$$

bbb Verification

$$\begin{aligned} &\dot{x}_1 = 1 \quad \& \quad \ddot{x}_1 = 0 \qquad x_1 \in R^1 \quad \& \quad x_1 \neq 0 \\ &\dot{x}_2 = (\sin x_1 - x_2)/x_1 \qquad x_2 \in R^1 \end{aligned} \tag{26}$$

Then

$$\ddot{x}_2 = [\dot{x}_1(x_2 - \sin \dot{x}_1 + x_1 \cos x_1) - x_1 \dot{x}_2]/x_1^2 \tag{27}$$

Applying the condition for the t-invariant ODE

$$\frac{\partial F(\mathbf{x})}{\partial \mathbf{x}}\ddot{\mathbf{x}} = (\sin x - y)\ddot{x} - x\ddot{y} = 0 \tag{28}$$

$$\begin{aligned} \frac{\partial F(\mathbf{x})}{\partial \mathbf{x}}\ddot{\mathbf{x}} &= \frac{x_1[\dot{x}_1(x_2 - \sin \dot{x}_1 + x_1 \cos x_1) - x_1 \dot{x}_2]}{x_1^2} \\ &= \frac{\dot{x}_1(x_2 - \sin \dot{x}_1 + x_1 \cos x_1) - x_1 \dot{x}_2}{x_1} \\ &= \frac{\dot{x}_1(x_2 - \sin \dot{x}_1)}{x_1} + \cos x_1 - \dot{x}_2 = 0 \end{aligned} \tag{29}$$

As $\dot{x}_1 = 1$, then

$$\begin{aligned} \frac{\partial F(\mathbf{x})}{\partial \mathbf{x}}\ddot{\mathbf{x}} &= x_2 + \cos(1) - \sin(1) - \dot{x}_2 = 0 \\ \frac{\partial F(\mathbf{x})}{\partial \mathbf{x}}\ddot{\mathbf{x}} &= x_2 - (\sin(1) - \cos(1)) - \dot{x}_2 = 0 \end{aligned} \tag{30}$$

Using identity

$$\sin \alpha \pm \cos \beta = 2 \sin(\frac{\alpha \pm \beta}{2}) \cos(\frac{\alpha \mp \beta}{2}) \tag{31}$$

$$\frac{\partial F(\mathbf{x})}{\partial \mathbf{x}}\ddot{\mathbf{x}} = x_2 - 2 \sin(1). \cos(0) - \dot{x}_2 = 0 \tag{32}$$

As $\dot{x}_2 = 0$ for any critical points, the new condition for critical points of t-invariant ordinary differential equation is obtained

$$\frac{\partial F(\mathbf{x})}{\partial \mathbf{x}}\ddot{\mathbf{x}} = 0 \tag{33}$$

It can be seen, that there is a formal connection to the Frenet-Serret formula, see Kabel[9], Kim[10] and WiKi[21].

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5 Conclusion

This paper briefly describes a new condition for critical points of ordinary differential equations using the Taylor expansion, see Eq.21. This condition increases robustness of the critical points detection especially in the case of discrete acquired data.

A general form for t-varying differential equations is derived and specification for the t-invariant differential equations is presented. A simple example demonstrating the approach is given, too.

In future, the proposed approach is to be applied for more complex cases, when physical phenomena is described by the ordinary differential equations, e.g. Skala[15], Smolik[18], etc. using Taylor expansion for vector data Skala[14] without tensor representation, and in applications described by partial differential equations, e.g Biancolini[2]

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