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A New Approach to Vector Field Interpolation, Classification and Robust Critical Points Detection using Radial Basis Functions

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Abstract. Visualization of vector fields plays an important role in many applications. Vector fields can be described by differential equations. For classification null points, i.e. points where derivation is zero, are used. However, if vector field data are given in a discrete form, e.g. by data obtained by simulation or a measurement, finding of critical points is difficult due to huge amount of data to be processed and differential form usually used. This contribution describes a new approach for vector field null points detection and evaluation, which enables data compression and easier fundamental behavior visualization. The approach is based on implicit form representation of vector fields.

Keywords: critical points, vector field classification, vector field topology, approximation, data acquisition, visualization, radial basis functions, RBF, interpolation, approximation.

1 Introduction

Many physical problems are described by differential equations of three basic types: ordinary differential equations (ODEs), partial differential equations (PDEs), algebraic-differential equations (ADEs or DAEs). They also can be classified as autonomous or t-varying, i.e. when functions depend on time. In this contribution, vector fields of autonomous system ODEs will be explored.

Let us imagine that a differential equation is given in E^2 as

$$\dot{x} = \boldsymbol{f}(\boldsymbol{x}(t), t) \tag{1}$$

where $\boldsymbol{f}(\boldsymbol{x}, t) = \left[{}^{x}f(\boldsymbol{x}, t),\ {}^{y}f(\boldsymbol{x}, t) \right]^T$. Implicit formulation is given as

$$F(\boldsymbol{x}(t), t) = 0 \tag{2}$$

where $t \in \langle 0,\infty)$, $\boldsymbol{x} \in E^2$. Derivation of the Eq.2. leads to

$$\frac{dF(x,t)}{dt} = \frac{\partial F(x,t)}{\partial x}\frac{dx}{dt} + \frac{\partial F(x,t)}{\partial t} \tag{3}$$

As the only autonomous ODEs are considered, i.e. $\partial F(\boldsymbol{x},t)/\partial t = 0$,

$$\frac{dF(x,t)}{dt} = \frac{\partial F(x,t)}{\partial x}\frac{dx}{dt} = \nabla F(\boldsymbol{x}(t))\frac{dx}{dt} = \nabla F(\boldsymbol{x}(t))\boldsymbol{f}(\boldsymbol{x}(t)) = \nabla F\dot{\boldsymbol{x}} \tag{4}$$

It means that a normal vector must be orthogonal to the particle velocity vector.

2 Extremes and inflection points

An inflection point of a curve given by the implicit function $F(x,y) = 0$ in E^2 is determined as $\det \boldsymbol{Q} = 0$, i.e.

$$\det \boldsymbol{Q}(x,y) = \begin{vmatrix} F_{xx} & F_{xy} & F_x \\ F_{yx} & F_{yy} & F_y \\ F_x & F_y & 0 \end{vmatrix} = 0 \tag{5}$$

where F_{xx}, resp. ${}^{x}f_y$ etc. are partial derivatives of $F(\boldsymbol{x})$, resp. ${}^{x}f_y = \frac{\partial\, {}^{x}f}{\partial y}$, etc. In the following, we expect that $F_{xy} = F_{yx}$. Details and extensions to a higher dimension can be found in (Goldman, 2005).

3 Critical points

The aim is to represent discreetly given vector field using Radial Basis Function (RBF) approximation as precise as possible in the form

$${}^{x}f(\boldsymbol{x}) = \sum_{i=1}^{N} c_i\, \phi(r_i) \tag{6}$$

where c_i are weights to be computed, $\phi(\cdot)$ is chosen RBF, e.g. $\phi(\cdot) = r^2 \log r$, $r_i = \|\boldsymbol{x} - \boldsymbol{x}_i\|$ and $\boldsymbol{x}_i$ are points in which the particle speed is given or acquired. Similarly for the ${}^{y}f(\boldsymbol{x})$. The advantage of the RBF use is that it leads to a linear system of equations $\boldsymbol{Ax} = \boldsymbol{b}$ (Smolik & Skala, 2017), (Majdisova & Skala, 2017).

Critical points of ODEs (or null points) are defined as $d\boldsymbol{x}(t)/dt = \boldsymbol{0}$. Finding of critical points of ODEs reliably is difficult and the Taylor's series is used usually, i.e.

$$\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{f}(\boldsymbol{x}_0) + \frac{\partial f(x_0)}{\partial x}(\boldsymbol{x} - \boldsymbol{x}_0) + \cdots \tag{7}$$

where $\boldsymbol{x}_0$ is a point where $\boldsymbol{f}(\boldsymbol{x}(t)) = \boldsymbol{0}$ and $\frac{\partial f(x_0)}{\partial x}$ is a Jacobian. It means that a local linearization is made and critical point classification is based on eigenvalues of the Jacobian (Helman & Hesselink, 1989). However for detailed inspection of a vector field a Hessian can be used (Smolik & Skala, 2017). Finding a null point of the ODE from acquired discrete data is a numerically sensitive problem.

Let us define a function $F(x,y)$ related to speed of a particle as:

$$F(x,y) = \dot{x}^2 + \dot{y}^2 \tag{8}$$

Then the critical points are given as:

$$\lim_{\varepsilon \to 0} F(x,y) - \varepsilon = 0 \tag{9}$$

The inflection points including the critical ones are given by $\det \boldsymbol{Q} = 0$.

Let us consider two simple ODEs examples, Fig.1., Fig.2., where critical points are shown. The example Ex.1 has two critical points, while the Ex.2 has three ones.

Ex.1.: $\begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix} = \begin{bmatrix}2x + y^2 - 1\\ 6x - y^2 + 1\end{bmatrix}$ Ex.2.: $\begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix} = \begin{bmatrix}xy - 4\\ (x-4)(y-x)\end{bmatrix}$

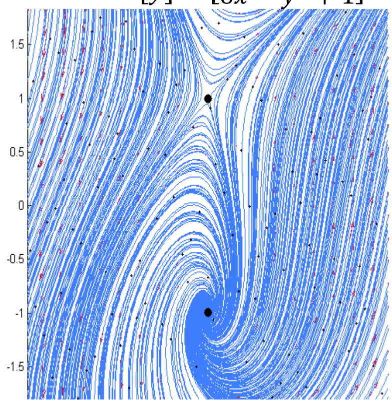

Fig. 1. Vector field with two critical points

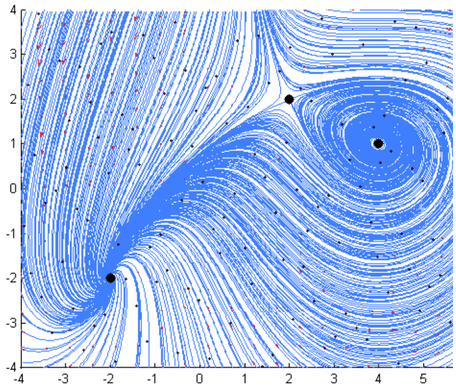

Fig. 2. Vector field with three critical points

When the contour plot of det $\boldsymbol{Q}$ values is made, some other interesting features of the given vector field can be found, Fig.3., Fig.4.

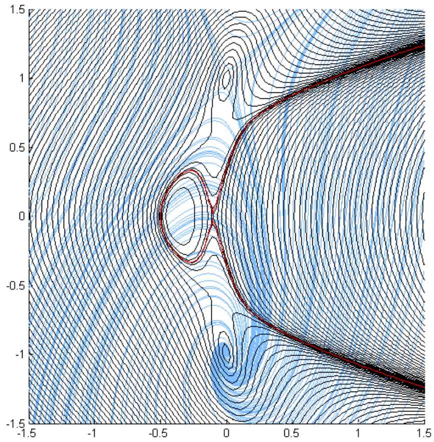

Fig. 3. $\det(\boldsymbol{Q})$ values for the Ex.1 case

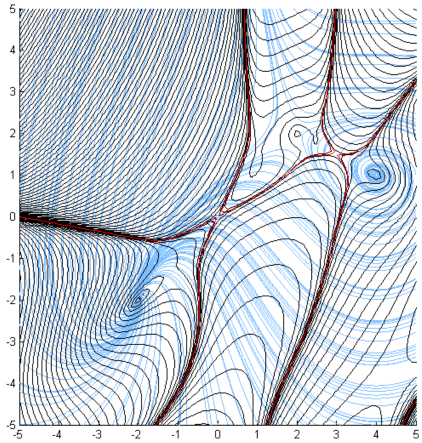

Fig. 4. $\det(\boldsymbol{Q})$ values for the Ex.2 case

It can be seen, that also other important points/areas, not only critical points, can be easily detected. The *red* curves represent points where $\det(\boldsymbol{Q}) = 0$, i.e. extremes and inflection points. The density of contours gives information on changes. If the RBF approximation of a vector field is to be efficiently used, reference points (Majdisova & Skala, 2017) are to be placed respecting the vector field behavior.

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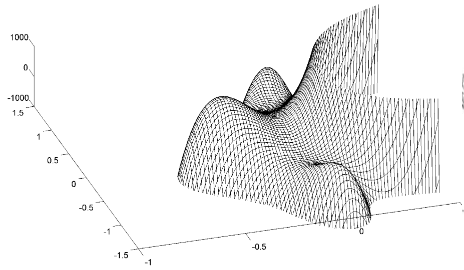

Fig. 5. $\det(\boldsymbol{Q})$ values for the Ex.1 case

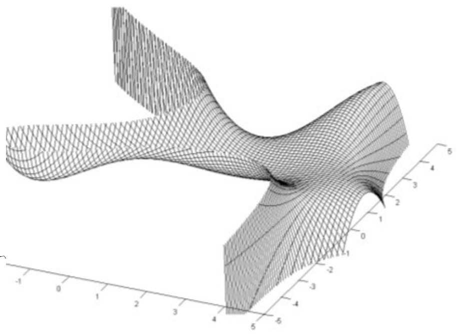

Fig. 6. $\det(\boldsymbol{Q})$ values for the Ex.2 case

It can be seen that in some cases the $\det \boldsymbol{Q}(x, y)$ function might be quite flat in some areas, which might cause some numerical problems. For better vector field properties evaluation, another specification of the $F(x, y)$ function using a similar approach, i.e.:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} {}^{x}f(x, y) \\ {}^{y}f(x, y) \end{bmatrix} \tag{10}$$

then the $\det \boldsymbol{R}$ is determined as:

$$\det \boldsymbol{R}(x, y) = \begin{vmatrix} {}^{x}f_x & {}^{x}f_y & {}^{x}f \\ {}^{y}f_x & {}^{y}f_y & {}^{y}f \\ {}^{x}f & {}^{y}f & 0 \end{vmatrix} = \begin{vmatrix} \boldsymbol{J}(\boldsymbol{x}) & \boldsymbol{f}(\boldsymbol{x}) \\ \boldsymbol{f}^T(\boldsymbol{x}) & 0 \end{vmatrix} = 0 \tag{11}$$

where $\boldsymbol{J}(\boldsymbol{x})$ is the Jacobian $\boldsymbol{J}(\boldsymbol{x}) = \begin{bmatrix} {}^{x}f_x & {}^{x}f_y \\ {}^{x}f_y & {}^{y}f_y \end{bmatrix}$ and $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$ is the given ODE.

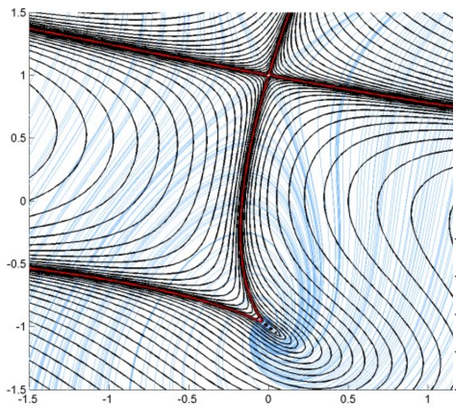

Fig. 7. $\det(\boldsymbol{R})$ values for the Ex.1 case

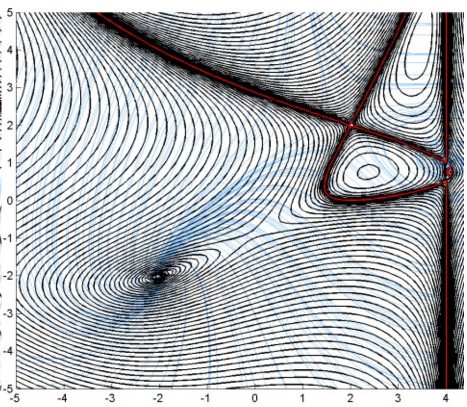

Fig. 8. $\det(\boldsymbol{R})$ values for the Ex.2 case

It can be seen, that another types of vector field important features are obtained. This is important for the vector field topology evaluation and curvatures estimation.

Ex.3.: $\begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix} = \begin{bmatrix}1 - x^2 - y^2\\ 2xy\end{bmatrix}$

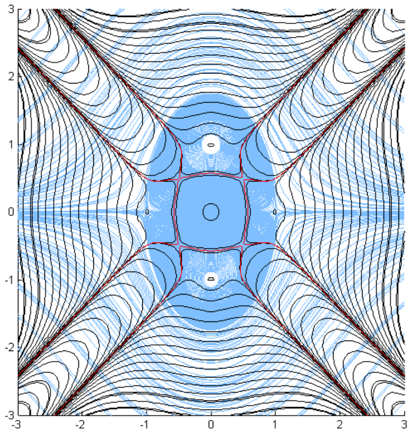

Fig. 9. $\det(\boldsymbol{R})$ values for the Ex.3 case

Ex.4.: $\begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix} = \begin{bmatrix}y\\ (1 - x^2)y - x\end{bmatrix}$

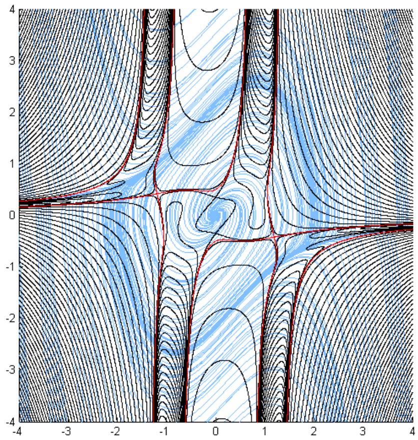

Fig. 10. $\det(\boldsymbol{R})$ values for the Ex.4 case (Van der Pole ODEs)

The *red* curves in Fig.9, Fig.10 represent the inflection of curves given as $\det \boldsymbol{Q} = 0$.

However, in some cases critical point finding using Eq.9 might be a numerical problem, if the function bed is too flat. Then the criterion can be modified to

$$F(x, y) = \sqrt{\dot{x}^2 + \dot{y}^2} \qquad (12)$$

In this case, the convergence for iterative method might be significantly faster and critical points more accurately determined.

4 Experimental evaluation

The proposed approach has been tested on different ODEs. Numerical computation of derivatives (forward, central, backward) was stable and critical points were determined correctly. The experiments proved novelty of the approach as an additional information of a vector field behavior is obtained. It can be used for efficient placing of reference points for RBF approximation (Majdisova & Skala, 2017). Also, if the RBF approximation is used, then the given vector field is described in an analytic form.

5 Conclusion

A new approach for vector field null points, i.e. critical points, is described briefly. The approach is based on an implicit formulation and it gives possibility to represent main features of vector fields more precisely. It also, in connection with RBF approximation, offers analytical representation of vector fields and data compression as well. In future, the presented approach will be extended to time varying systems.

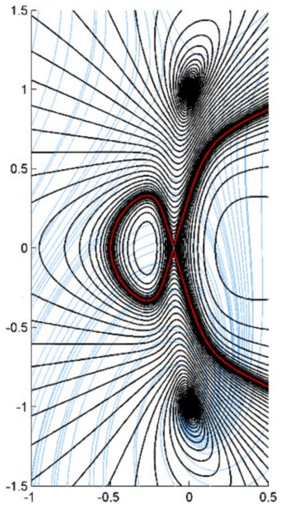

Fig. 11. Ex.3. $\det(\boldsymbol{Q})$ using Eq.12

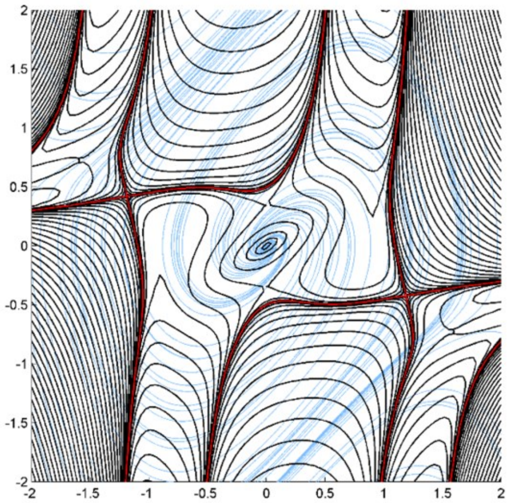

Fig. 12. Ex.4. $\det(\boldsymbol{Q})$ using Eq.12

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