Vector Field Second Order Derivative Approximation and Geometrical Characteristics

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Abstract. Vector field is mostly linearly approximated for the purpose of classification and description. This approximation gives us only basic information of the vector field. We will show how to approximate the vector field with second order derivatives, i.e. Hessian and Jacobian matrices. This approximation gives us much more detailed description of the vector field. Moreover, we will show the similarity of this approximation with conic section formula.

Keywords: Vector field; Critical point; Geometry; Conic section; Hessian matrix.

1 Introduction

The visualization of vector field topology is a problem that arises naturally when studying the qualitative structure of flows that are tangential to some surface. The knowledge of the data in a single point would be of little help when the goal is to obtain knowledge and understanding of the whole vector field. The individual numbers can be of little interest. It is the connection between them, which is important.

Helman and Hesselink [6] introduced the concept of the topology of a planar vector field to the visualization community. They extracted critical points and classified them into sources, sinks and saddles, and integrated certain stream lines called separatrices from the saddles in the directions of the eigenvectors of the Jacobian matrix. Later, topological methods have been extended to higher order critical points [14], boundary switch points [10], and closed separatrices [21]. In addition, topological methods using classification have been applied to simplify [16], [15], smooth [20], compress [1], [7], compare [11] and design vector fields.

The published research methods use for classification of critical points and vector field description only linear approximation of the vector field. None of it uses an approximation with second order partial derivatives, i.e. Hessian matrix. This approximation gives a more detailed description of the vector field around a critical point and can be used for a more detailed classification. Use of the approximation with Hessian matrix will be described in this paper.

2 Vector Field Approximation

Vector fields [18] on surfaces [17] are important objects, which appear frequently in scientific simulation in CFD (Computational Fluid Dynamics) [2], [12] or modelling by FEM (Finite Element Method). To be visualized [5], [8], such vector fields are usually linearly approximated for the sake of simplicity and performance considerations. Other possible approximations are [3], [4], [9].

The vector field can be easily analyzed when having an approximation of the vector field near some location point. The important places to be analyzed are so called critical points. Analyzing the vector field behavior near these points gives us the information about the characteristic of the vector field.

2.1 Critical Point

Critical points ($\boldsymbol{x}_0$) of the vector field are points at which the magnitude of the vector vanishes

$$\frac{d\boldsymbol{x}}{dt} = \boldsymbol{v}(\boldsymbol{x}) = \boldsymbol{0}\,, \tag{1}$$

i.e. all components are equal to zero

$$\begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{2}$$

A critical point is said to be isolated, or simple, if the vector field is non-vanishing in an open neighborhood around the critical point. Thus for all surrounding points $\boldsymbol{x}_\varepsilon$ of the critical point $\boldsymbol{x}_0$ the equation (1) does not apply, i.e.

$$\frac{d\boldsymbol{x}_\varepsilon}{dt} \neq \boldsymbol{0}\,. \tag{3}$$

At critical points, the direction of the field line is indeterminate, and they are the only points in the vector field where field lines can intersect (asymptotically). The terms singular point, null point, neutral point or equilibrium point are also frequently used to describe critical points.

These points are important because together with the nearby surrounding vectors, they have more information encoded in them than any such group in the vector field, regarding the total behavior of the field.

2.2 Linearization of Vector Field

Critical points can be characterized according to the behavior of nearby tangent curves. We can use a particular set of these curves to define a skeleton that characterizes the global behavior of all other tangent curves in the vector field. An important feature of differential equations is that it is often possible to determine the local stability of a critical point by approximating the system by a linear system. These approximations are aimed at studying the local behavior of a system, where the nonlinear effects are expected to be small. To locally approximate a system, the Taylor series expansion must be utilized locally to find the relation between $\boldsymbol{v}$ and position $\boldsymbol{x}$, supposing the flow $\boldsymbol{v}$ to be sufficiently smooth and differentiable. In such case, the expansion of $\boldsymbol{v}$ around the critical points $\boldsymbol{x}_0$ is

$$\boldsymbol{v}(\boldsymbol{x}) = \boldsymbol{v}(\boldsymbol{x}_0) + \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{x}}(\boldsymbol{x} - \boldsymbol{x}_0)\,. \tag{4}$$

As $\boldsymbol{v}(\boldsymbol{x}_0)$ is according to (1) equal zero for critical points, we can rewrite equation (4) using matrix notation

$$\begin{bmatrix} v_x \\ v_y \end{bmatrix} = \begin{bmatrix} \frac{\partial v_x}{\partial x} & \frac{\partial v_x}{\partial y} \\ \frac{\partial v_y}{\partial x} & \frac{\partial v_y}{\partial y} \end{bmatrix} \cdot \begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix} \tag{5}$$

$$\boldsymbol{v} = \boldsymbol{J} \cdot (\boldsymbol{x} - \boldsymbol{x}_0)\,, \tag{6}$$

where $\boldsymbol{J}$ is called Jacobian matrix and characterizes the vector field behavior around a critical point $\boldsymbol{x}_0$.

2.3 Approximation Using Hessian Matrix

Vector fields are approximated using only linear approximation to determine the local behavior of the vector field. However, linearization gives as basic classification of the critical points and about the flow around them, the approximation using second order derivatives will give us some more information.

The approximation of vector field around a critical point using the second order derivative must be written for each vector component (v_x and v_y) separately, see the following equations

$$v_x = \begin{bmatrix} \frac{\partial v_x}{\partial x} \\ \frac{\partial v_x}{\partial y} \end{bmatrix}^T \cdot \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} + \frac{1}{2} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix}^T \cdot \begin{bmatrix} \frac{\partial^2 v_x}{\partial x^2} & \frac{\partial^2 v_x}{\partial x \partial y} \\ \frac{\partial^2 v_x}{\partial y \partial x} & \frac{\partial^2 v_x}{\partial y^2} \end{bmatrix} \cdot \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} \tag{7}$$

$$v_y = \begin{bmatrix} \frac{\partial v_y}{\partial x} \\ \frac{\partial v_y}{\partial y} \end{bmatrix}^T \cdot \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} + \frac{1}{2} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix}^T \cdot \begin{bmatrix} \frac{\partial^2 v_y}{\partial x^2} & \frac{\partial^2 v_y}{\partial x \partial y} \\ \frac{\partial^2 v_y}{\partial y \partial x} & \frac{\partial^2 v_y}{\partial y^2} \end{bmatrix} \cdot \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix}, \tag{8}$$

where $\Delta x = x - x_0$ and $\Delta y = y - y_0$. These two equations can be written in matrix notation as well

$$v_x = \boldsymbol{J}_x \cdot (\boldsymbol{x} - \boldsymbol{x}_0) + \frac{1}{2}(\boldsymbol{x} - \boldsymbol{x}_0)^T \cdot \boldsymbol{H}_x \cdot (\boldsymbol{x} - \boldsymbol{x}_0) \tag{9}$$

$$v_y = \boldsymbol{J}_y \cdot (\boldsymbol{x} - \boldsymbol{x}_0) + \frac{1}{2}(\boldsymbol{x} - \boldsymbol{x}_0)^T \cdot \boldsymbol{H}_y \cdot (\boldsymbol{x} - \boldsymbol{x}_0)\,, \tag{10}$$

where $\boldsymbol{H}_x$ and $\boldsymbol{H}_y$ are Hessian matrices, $\boldsymbol{J}_x$ is the first row of Jacobian matrix and $\boldsymbol{J}_y$ is the second row of Jacobian matrix.

The Hessian matrix is a square matrix of second-order partial derivatives of a scalar-valued function, or scalar field. It describes the local curvature of a function of many variables.

Approximation of vector field using (7) and (8) gives us more detailed description than approximation of vector field using (5), see Fig. 1. The approximation in Fig. 1 (right) gives us the same information like in Fig. 1 (left), although we can see the curvature of the two main axis for the saddle.

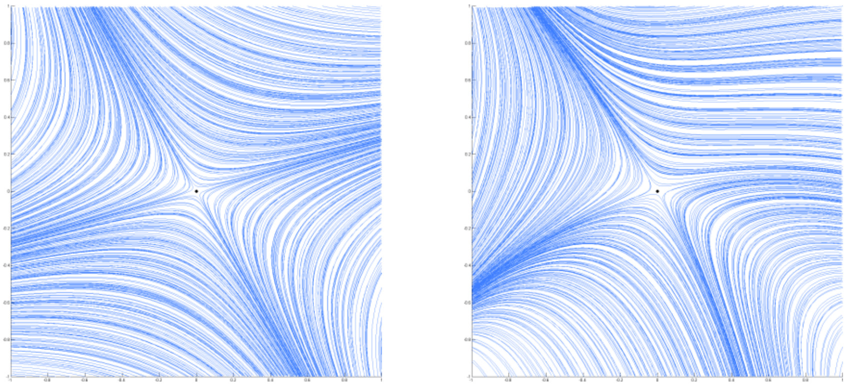

Fig. 1. Comparison between the phase portraits for the vector field approximated using linear approximation (left) and using second order derivative (right).

Equations (7) and (8) can be rewritten in different formulas as follows

$$v_x = \frac{1}{2}[\Delta x \quad \Delta y \quad 1] \cdot \begin{bmatrix} \frac{\partial^2 v_x}{\partial x^2} & \frac{\partial^2 v_x}{\partial x \partial y} & \frac{\partial v_x}{\partial x} \\ \frac{\partial^2 v_x}{\partial y \partial x} & \frac{\partial^2 v_x}{\partial y^2} & \frac{\partial v_x}{\partial y} \\ \frac{\partial v_x}{\partial x} & \frac{\partial v_x}{\partial y} & 0 \end{bmatrix} \cdot \begin{bmatrix} \Delta x \\ \Delta y \\ 1 \end{bmatrix} \tag{11}$$

$$v_y = \frac{1}{2}[\Delta x \quad \Delta y \quad 1] \cdot \begin{bmatrix} \frac{\partial^2 v_y}{\partial x^2} & \frac{\partial^2 v_y}{\partial x \partial y} & \frac{\partial v_y}{\partial x} \\ \frac{\partial^2 v_y}{\partial y \partial x} & \frac{\partial^2 v_y}{\partial y^2} & \frac{\partial v_y}{\partial y} \\ \frac{\partial v_y}{\partial x} & \frac{\partial v_y}{\partial y} & 0 \end{bmatrix} \cdot \begin{bmatrix} \Delta x \\ \Delta y \\ 1 \end{bmatrix}. \qquad (12)$$

These two equations have some geometrical background. When v_x and v_y are equal zero, each equation describes some conic section.

Approximation of the vector field using Hessian matrix, i.e. using second order derivatives, is a bit more computationally expensive than the standard linear approximation but gives us more detailed description of the vector field as will be seen in the following chapters.

Conic Section.

A conic is the curve obtained as the intersection of a plane, called the cutting plane, with a double cone, see Fig. 2. Planes that pass through the vertex of the cone will intersect the cone in a point, a line or a pair of intersecting lines. These are called degenerate conics and some authors do not consider them to be conics at all.

There are three types of non-degenerated conics, the ellipse, parabola, and hyperbola**,** see Fig. 2. The circle is a special kind of ellipse. The circle and the ellipse arise when the intersection of the cone and plane is a closed curve. The circle is obtained when the cutting plane is parallel to the plane of the generating circle of the cone, this means that the cutting plane is perpendicular to the symmetry axis of the cone. If the cutting plane is parallel to exactly one generating line of the cone, then the conic is unbounded and is called a parabola. In the remaining case, the figure is a hyperbola. In this case, the plane will intersect both halves of the cone, producing two separate unbounded curves.

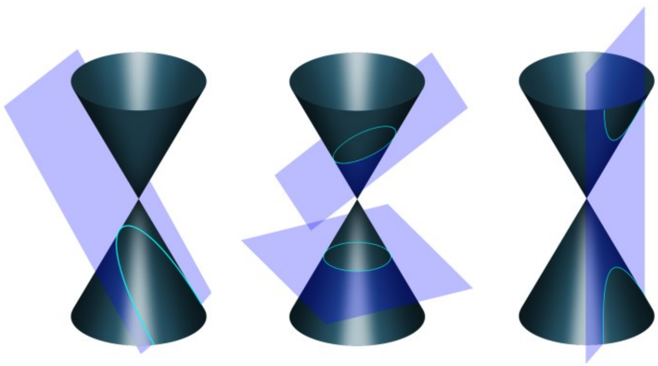

Fig. 2. Types of conic sections, i.e. parabola, circle and ellipse, and hyperbola.

A conic section is described by the following implicit equation

$$[x \quad y \quad 1] \cdot \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = 0\,. \tag{13}$$

where a_{ij} $i, j \in \{1, 2, 3\}$ are coefficients of conic section. Depending on these values, we can classify the types of conic sections. To do that, we need to compute two determinants

$$\Omega = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \tag{14}$$

$$\omega = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}\,. \tag{15}$$

When knowing determinants Ω and ω we can easily classify the type of conic section using the following table

Table 1: Classification of conic section.

	$\omega \neq 0$		$\omega = 0$
$\Omega \neq 0$	$\omega > 0$	$\omega < 0$	parabola
	ellipse	hyperbola	
$\Omega = 0$	pair of intersecting lines		pair of parallel lines

Equations (11) and (12) are the same as (13) when $v_x = 0$ and $v_y = 0$ and therefore they geometrically represent conic sections.

3 Classification of Critical Points

There exist a finite set of fundamentally different critical points, defined by the number of inflow and outflow directions, spiraling structures etc., and combinations of these. Since the set is finite, each critical point can be classified. Such a classification defines the field completely in a close neighborhood around the critical point. By knowing the location and classification of critical points in a vector field, the topology of the field is known in small areas around these. Assuming a smooth transition between these areas, one can construct a simplified model of the whole vector field. Such a simplified representation is useful, for instance, in compressing vector field data into simpler building blocks [13].

The critical points are classified based on the vector field around that point. The information derived from the classification of critical points aids the information selection process when it comes to visualizing the field. By choosing seed points for field lines based on the topology of critical points, field lines encoding important information is ensured. A more advanced approach is to connect critical points, and use the connecting lines and surfaces to separate areas of different flow topology [1], [19].

3.1 Standard Classification Using a Linear Approximation

The fact that a linear model can be used to study the behavior of a nonlinear system near a critical point is a powerful one [1]. We can use the Jacobian matrix to characterize the vector field and the behavior of nearby tangent curves, for nondegenerate critical point.

The eigenvalues and eigenvectors of Jacobian matrix are very important for vector field classification and description (see Fig. 3). A real eigenvector of the Jacobian matrix defines a direction such that if we move slightly from the critical point in that direction, the field is parallel to the direction we moved. Thus, at the critical point, the real eigenvectors are tangent to the trajectories that end on the point. The sign of the corresponding eigenvalue determines whether the trajectory is outgoing (repelling) or incoming (attracting) at the critical point. The imaginary part of an eigenvalue denotes circulation about the point.

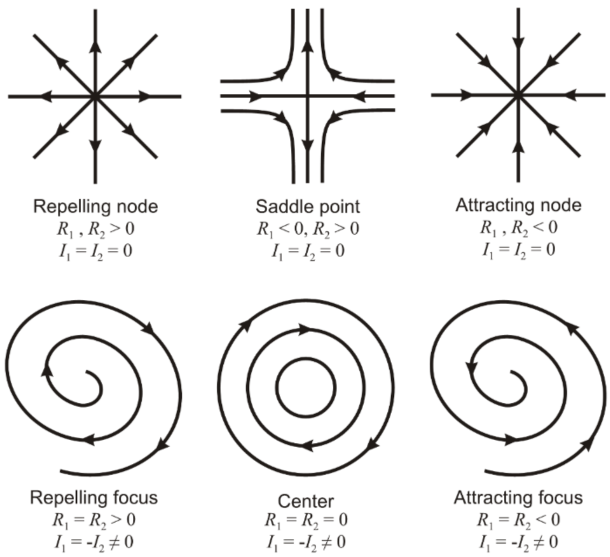

Fig. 3. Classification of $2D$ first order critical points. R_1, R_2 denote the real parts of the eigenvalues of the Jacobian matrix while I_1, I_2 denote their imaginary parts (from [1]).

3.2 Classification Using Description of Conic Sections

Each vector field can be approximated at a critical point with the approximation that uses the second order derivatives, i.e. Hessian matrix. One such example of approximated vector field around a critical point $\boldsymbol{x}_0 = [0, 0]^T$ can be

$$v_x = \frac{1}{2}[\Delta x \quad \Delta y \quad 1] \cdot \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 2 \\ 1 & 2 & 0 \end{bmatrix} \cdot \begin{bmatrix} \Delta x \\ \Delta y \\ 1 \end{bmatrix}, \tag{16}$$

$$v_y = \frac{1}{2}[\Delta x \quad \Delta y \quad 1] \cdot \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 1.5 \\ -1 & 1.5 & 0 \end{bmatrix} \cdot \begin{bmatrix} \Delta x \\ \Delta y \\ 1 \end{bmatrix}. \tag{17}$$

Equation (16) represents for $v_x = 0$ a parabola and (17) for $v_y = 0$ a line. This approximated vector field can be seen in Fig. 4.

Now, we showed conic sections that have only one intersection point at $[0, 0]^T$. Two conic sections can have up to four intersections. Each intersection defines a critical point. Therefore, we can approximate a vector field around one critical point and some more critical points in the neighborhood will be included in this approximation.

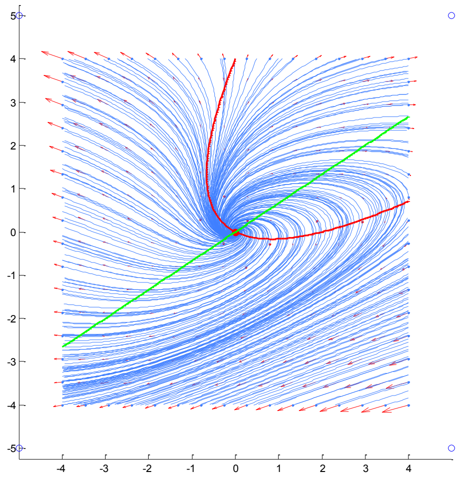

Fig. 4. Vector field approximated as (16) and (17). The zero iso-lines are a line and a parabola.

Vector fields around a focus critical point can be for some real vector field approximated for example as

$$\begin{aligned} v_x &= \frac{1}{2}[\Delta x \;\; \Delta y \;\; 1] \cdot \begin{bmatrix} 1 & -3 & 1 \\ -3 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix} \cdot \begin{bmatrix} \Delta x \\ \Delta y \\ 1 \end{bmatrix} \\ v_y &= \frac{1}{2}[\Delta x \;\; \Delta y \;\; 1] \cdot \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 1.5 \\ -1 & 1.5 & 0 \end{bmatrix} \cdot \begin{bmatrix} \Delta x \\ \Delta y \\ 1 \end{bmatrix} \end{aligned} \tag{18}$$

$$v_x = \frac{1}{2}[\Delta x \;\; \Delta y \;\; 1] \cdot \begin{bmatrix} -0.5 & 0.5 & 1 \\ 0.5 & -0.5 & 2 \\ 1 & 2 & 0 \end{bmatrix} \cdot \begin{bmatrix} \Delta x \\ \Delta y \\ 1 \end{bmatrix}$$

$$v_y = \frac{1}{2}[\Delta x \;\; \Delta y \;\; 1] \cdot \begin{bmatrix} -1 & 1 & -1 \\ 1 & -1 & 1.5 \\ -1 & 1.5 & 0 \end{bmatrix} \cdot \begin{bmatrix} \Delta x \\ \Delta y \\ 1 \end{bmatrix} \quad (19)$$

This both approximations of vector fields describe behavior around a focus critical point at $[0,0]^T$. Both of them contain one more critical point, which is a saddle critical point. These saddle critical points do not have to be real critical points of the approximated vector field, but they can be present in the vector field. Therefore, this approximation can give us some information about other possible critical points in the neighborhood of approximated critical point $\boldsymbol{x}_0$. When locating all critical points in the vector field, we can use this information to increase the probability of finding all critical points.

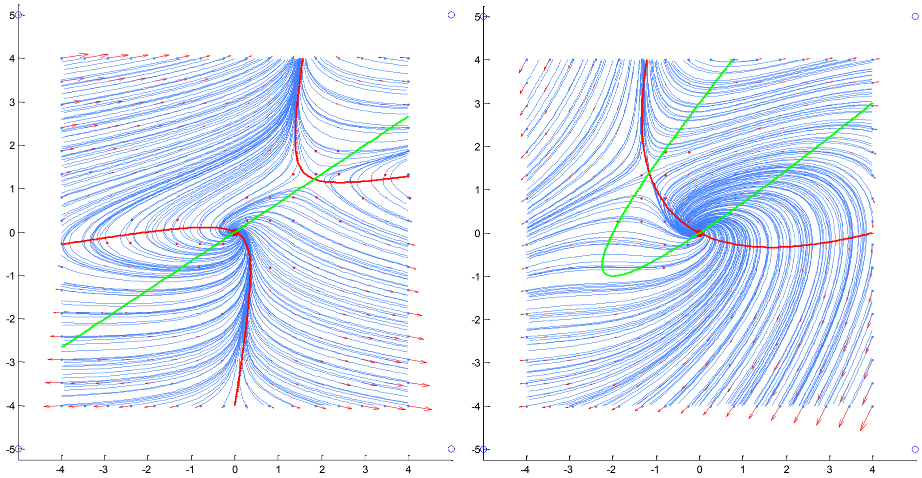

Fig. 5. Vector field approximated as (18) (left) and (19) (right). The zero iso-lines are a line and a hyperbola (left), or two parabolas (right).

The maximal number of two conic sections intersection points is four. In the next example, we will show it. Let us have a vector field, which can be approximated at point $\boldsymbol{x}_0$ for example as

$$v_x = \frac{1}{2}[\Delta x \;\; \Delta y \;\; 1] \cdot \begin{bmatrix} -0.25 & 0 & 1 \\ 0 & -1 & 2 \\ 1 & 2 & 0 \end{bmatrix} \cdot \begin{bmatrix} \Delta x \\ \Delta y \\ 1 \end{bmatrix}$$

$$v_y = \frac{1}{2}[\Delta x \;\; \Delta y \;\; 1] \cdot \begin{bmatrix} -1 & 1 & -1 \\ 1 & -1 & 1.5 \\ -1 & 1.5 & 0 \end{bmatrix} \cdot \begin{bmatrix} \Delta x \\ \Delta y \\ 1 \end{bmatrix} \quad (20)$$

$$v_x = \frac{1}{2}[\Delta x \;\; \Delta y \;\; 1] \cdot \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 0 \end{bmatrix} \cdot \begin{bmatrix} \Delta x \\ \Delta y \\ 1 \end{bmatrix}$$

$$v_y = \frac{1}{2}[\Delta x \;\; \Delta y \;\; 1] \cdot \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1.5 & 1.5 \\ -1 & 1.5 & 0 \end{bmatrix} \cdot \begin{bmatrix} \Delta x \\ \Delta y \\ 1 \end{bmatrix} \tag{21}$$

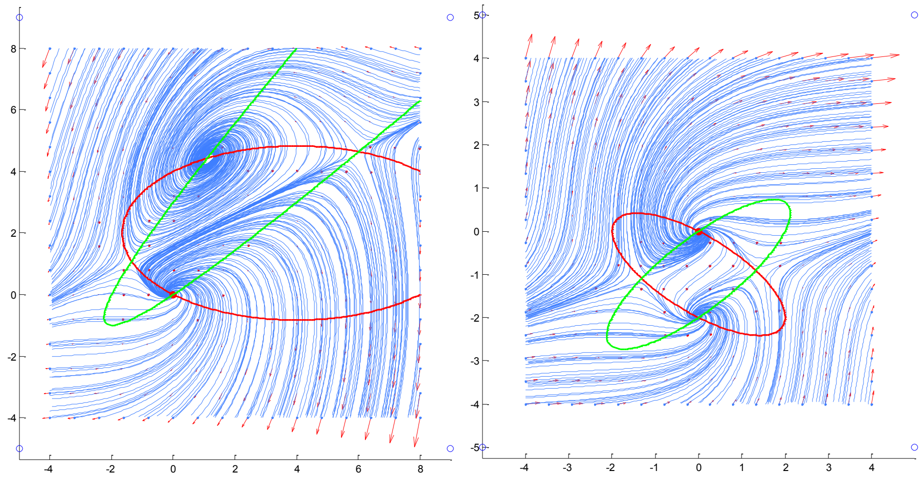

Fig. 6. Vector field approximated as (20) (left) and (21) (right). The zero iso-lines are a parabola and an ellipse (left), or two ellipses (right).

These two approximations (20) and (21) of vector fields are visualized in Fig. 6. It can be seen, that each approximation contains four critical points, i.e. one critical point where the vector field was approximated and three more critical points.

4 Conclusion

A new vector field critical points description using the second order derivatives approximation is described. The approximation can be rewritten in a matrix form of a conic section formula. We proved, that approximation using Hessian matrix, rather than only Jacobian matrix, gives us better representation of a vector field and it can help with localization of critical points in a vector field.

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