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Geometric Algebra, Extended Cross-product and Laplace Transform for Multidimensional Dynamical Systems

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Abstract. This contribution describes a new approach for solving linear system of algebraic equations and differential equations using Laplace transform by the extended-cross product. It will be shown that a solution of a linear system of equations Ax=0 or Ax=b is equivalent to the extended cross-product if the projective extension of the Euclidean system and the principle of duality are used. Using the Laplace transform differential equations are transformed to a system of linear algebraic equations, which can be solved using the extended cross-product (outer product). The presented approach enables to avoid division operation and extents numerical precision as well. It also offers applications of matrix-vector and vector-vector operations in symbolic manipulation, which can leads to new algorithms and/or new formula. The proposed approach can be applied also for stability evaluation of dynamical systems. In the case of numerical computation, it supports vector operation and SSE instructions or GPU can be used efficiently.

Keywords: Linear system of equations, linear system of differential equations, Laplace transform, extended cross product, outer product, homogeneous coordinates, duality, geometrical algebra, dynamic systems, stability, GPGPU computation, SSE instructions.

1 Introduction

Solving system of linear algebraic equations is often used in many applications. However, methods for solution differ if the linear system of equations is homogeneous, i.e. $\boldsymbol{Ax} = \boldsymbol{0}$, or non-homogeneous $\boldsymbol{Ax} = \boldsymbol{b}$. If the projective extension of the Euclidean space is used and principle of duality applied, the both cases can be solved using extended cross-product as $\boldsymbol{\alpha}_1 \times \boldsymbol{\alpha}_2 \times ... \times \boldsymbol{\alpha}_n$ or as $\boldsymbol{\alpha}_1 \wedge \boldsymbol{\alpha}_2 \wedge ... \wedge \boldsymbol{\alpha}_n$ if outer product is used, where $\boldsymbol{\alpha}_i$ is the i −th row of the matrix $\boldsymbol{A}$, resp. $[\boldsymbol{A}|-\boldsymbol{b}]$ [10]-[18].

In the case of differential equations, the Laplace transform transforms differential system to an algebraic system of equations. It can be seen that the extended cross-product does not use any division operation as would be expected in solution of a linear

adfa, p. 1, 2011.

system of equations. In addition, it means that standard vector and/or matrix operations can be applied in further processing and solution of the system of equations can be avoided in principle. Symbolic manipulations using vector notation might lead to better understanding and possibly to derive new formulas.

In the following the Laplace transform, duality and solution of algebraic linear system using extended cross-product will be shortly introduced.

2 Laplace Transformation

Pierre-Simon Laplace discovered the Laplace transform in 1785. It is an integral transform applied on a real function $f(t)$ with a real positive argument $t \geq 0$ and converts the function it to a complex function $F(s)$ with a complex argument $s = \delta + i\omega$. The Laplace transform is defined as:

$$\mathcal{L}\{f(t)\} = F(s) \qquad F(s) = \int_0^{\infty} f(t)e^{-st}dt \tag{1}$$

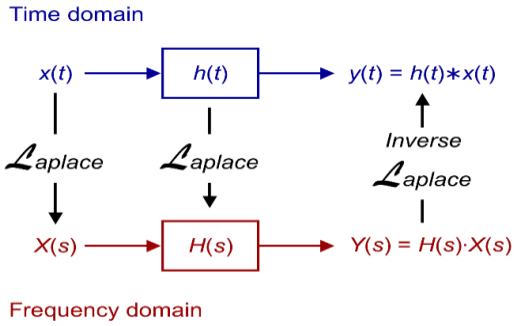

Fig.1.Taken from https://en.wikibooks.org [24]

The Laplace transform, see Fig.1, is often used for transformation of differential system of equations to algebraic equations and convolution to multiplication [3],[4].

TABLE I. TYPICAL LAPLACE TRANSFORM PATTERNS

Time domain	s domain
$f(t)$	$F(s)$
$af(t) + bg(t)$	$aF(s) + bG(s)$
$f'(t)$	$sF(s) - f(0)$
$f''(t)$	$s^2F(s) - sf(0) - f'(0)$
t	$1/s^2$
$f(0)$	$\lim_{s\to\infty} sF(s)$
$\lim_{t\to\infty} f(t)$	$\lim_{s\to 0} sF(s)$
$f(t) * g(t)$ (convolution)	$F(s)G(s)$

It means, that a system of differential equations is transformed to a system of linear equations, which is to be solved and then this solution is transformed back to the time domain using inverse Laplace transform; in many cases the result is decomposed to some "patterns" for which the inverse transform is known.

The solution is then transformed back to the time domain using the inverse Laplace transform.

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \frac{1}{2\pi i}\lim_{T\to\infty}\int_{\alpha-iT}^{\alpha+iT} F(s)e^{st}dt \tag{2}$$

where α is taken so that all singularities of $F(s)$ are on the left of $Re(s)$. In many cases the result is decomposed to some "patterns" for which the inverse transform is known.

Let us consider a simple example of a system of differential equations [22]:

$$x' = 3x - 3y + 2 \qquad y' = -6x - t \tag{3}$$

with initial conditions $x(0) = 1, y(0) = -1$. Applying the Laplace transform, we obtain a system of linear algebraic equations with respect to x, y as:

$$\begin{aligned} sX(s) - x(0) &= 3X(s) - 3Y(s) + \frac{2}{s} \\ sY(s) - y(0) &= -6X(s) - \frac{1}{s^2} \end{aligned} \tag{4}$$

Using algebraic manipulations:

$$\begin{aligned} (s-3)X(s) + 3Y(s) &= x(0) + \frac{2}{s} \\ 6X(s) + sY(s) &= y(0) - \frac{1}{s^2} \end{aligned} \tag{5}$$

If initial conditions are included, i.e. $x(0) = 1$ and $y(0) = -1$, it leads to a system of linear equations:

$$\begin{aligned} (s-3)X(s) + 3Y(s) &= 1 + \frac{2}{s} \\ 6X(s) + sY(s) &= -1 - \frac{1}{s^2} \end{aligned} \tag{6}$$

In the matrix form:

$$\begin{bmatrix} s-3 & 3 \\ 6 & s \end{bmatrix}\begin{bmatrix} X(s) \\ Y(s) \end{bmatrix} = \begin{bmatrix} \dfrac{s+2}{s} \\ -\dfrac{s^2+1}{s^2} \end{bmatrix} \tag{7}$$

Now, this system of linear equations is to be solved in order to obtain functions $X(s)$ and $Y(s)$ which are then transformed back to the time domain by the inverse Laplace transformation.

In the following, we introduce basic information geometric algebra, projective representation and principal of duality.

3 Geometric Algebra

The inner product is the most often algebraic construction in the n-dimensional Euclidean space. The Geometric Algebra (GA) is an inner product extension. The GA is not commutative and member of GA are called *multivectors.* The geometric product of two vectors in E^n is connected to the algebraic construction

$$\boldsymbol{uv} = \boldsymbol{u} \cdot \boldsymbol{v} + \boldsymbol{u} \wedge \boldsymbol{v} \tag{8}$$

where $\boldsymbol{uv}$ is the geometric product, $\boldsymbol{u} \cdot \boldsymbol{v}$ is the inner product and $\boldsymbol{u} \wedge \boldsymbol{v}$ is the outer product (in E^3 equivalent to the cross product, i.e. $\boldsymbol{u} \times \boldsymbol{v}$). If $\boldsymbol{e}_i$ are orthonormal basis vectors, then

$$\begin{aligned} 1 &\quad \text{0-vector (scalar)} \\ \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3 &\quad \text{1-vectors (vectors)} \\ \boldsymbol{e}_1\boldsymbol{e}_2, \boldsymbol{e}_2\boldsymbol{e}_3, \boldsymbol{e}_3\boldsymbol{e}_1 &\quad \text{2-vectors (bivectors)} \\ \boldsymbol{I} = \boldsymbol{e}_1\boldsymbol{e}_2\boldsymbol{e}_3 &\quad \text{3-vector (pseudoscalar)} \end{aligned} \tag{9}$$

It can be easily proved that the inner product is

$$\boldsymbol{u} \cdot \boldsymbol{v} = \frac{1}{2}(\boldsymbol{uv} + \boldsymbol{vu}) \tag{10}$$

There is something "strange" in the case of E^3 as the geometric product $\boldsymbol{uv} = \boldsymbol{u} \cdot \boldsymbol{v} + \boldsymbol{u} \wedge \boldsymbol{v}$ actually "accumulate" scalar value and result of the outer product, i.e. the cross product E^3, which is a *bivector*, actually not a vector. The size of it is an area of a rhomboid determined by the $\boldsymbol{u}, \boldsymbol{v}$ vectors the n-dimensional space in general. Due to the non-commutativity

$$\boldsymbol{u} \wedge \boldsymbol{v} = -\boldsymbol{v} \wedge \boldsymbol{u} \qquad \boldsymbol{uu} = \boldsymbol{u} \cdot \boldsymbol{u} = |\boldsymbol{u}|^2 \tag{11}$$

for all $\boldsymbol{u} \in R^n$. It means, that there is an inverse defined as

$$\boldsymbol{u}^{-1} = \boldsymbol{u}/_{|\boldsymbol{u}|^2} \tag{12}$$

There is another "object" called a *blade*. A *k-blade* $\boldsymbol{B}$ is a subspace given by orthogonal vectors $\boldsymbol{e}_{i_1}, \dots, \boldsymbol{e}_{i_k}$, where $\boldsymbol{e}_i \neq \boldsymbol{e}_j$. Similar operations with vectors, operations with *k-blades* are introduced [5][6][19].

4 Euclidean and Projective Spaces

The Euclidean space is used nearly exclusively in computational sciences. In some applications, like computer vision, computer graphics etc., the projective extension of the Euclidean space is used [2][9][20]. The projective extension in E^2 is defined as

$$X = \frac{x}{w} \qquad Y = \frac{y}{w} \qquad w \neq 0 \tag{13}$$

where x, y, w are homogeneous coordinates, i.e. $\boldsymbol{x} = [x, y: w]^T \in P^2$, $\boldsymbol{X} = (X, Y) \in E^2$ are coordinates in the Euclidean space. This concept is valid generally for the n-dimensional space. In general, a value in the projective space is represented as:

$$\boldsymbol{x} = [x_1, \dots, x_n: w]^T \in P^n \tag{14}$$

or as:

$$\boldsymbol{x} = [x_0 : x_1, \dots, x_n]^T \in P^n \tag{15}$$

where: x_0 stands for w; this notation is mostly used in mathematical resources. The symbol ":" means that the homogenous coordinate w is just a "scaling factor" and has no physical unit, while $x_1, \dots, x_n$ do have.

Let us introduce the extended cross product and its use with the projective space representation with simple geometrical examples for simplicity of explanation.

5 Extended Cross-product

The cross-product of two vectors $\boldsymbol{a}, \boldsymbol{b}$ in E^3 is defined:

$$\boldsymbol{q} = \boldsymbol{a} \times \boldsymbol{b} = \det \begin{bmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{bmatrix} \tag{16}$$

where: $\boldsymbol{i} = [1,0,0]^T$, $\boldsymbol{j} = [0,1,0]^T$, $\boldsymbol{k} = [0,0,1]^T$ are unit vectors. The result of the cross-product $\boldsymbol{q}$ is a "bivector" which is an oriented area of a rhomboid in E^3 given by the vectors $\boldsymbol{a}, \boldsymbol{b}$. It should not be handled as a "movable vector" in general [17][18].

The oriented area of the rhomboid given by three points in E^2 is determined as:

$$area = \det \begin{bmatrix} X_1 & Y_1 & 1 \\ X_2 & Y_2 & 1 \\ X_3 & Y_3 & 1 \end{bmatrix} \tag{17}$$

due to the linearity the rows can be multiplied as follows:

$$area = \frac{1}{w_1 w_2 w_3} \det \begin{bmatrix} w_1 X_1 & w_1 Y_1 & w_1 \\ w_2 X_2 & w_2 Y_2 & w_2 \\ w_3 X_3 & w_3 Y_3 & w_3 \end{bmatrix} = \frac{1}{w_1 w_2 w_3} \det \begin{bmatrix} x_1 & y_1 & w_1 \\ x_2 & y_2 & w_2 \\ x_3 & y_3 & w_3 \end{bmatrix} \tag{18}$$

It means that in some well-known formulas we do use projective representation. As another simple example, let us consider computation of the intersection point $\boldsymbol{X} = (X, Y)$ of two given lines p_1 and p_2 in E^2:

$$p_1 : a_1 X + b_1 Y + c_1 = 0 \qquad p_2 : a_2 X + b_2 Y + c_2 = 0 \tag{19}$$

It leads to a system of linear equations $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$:

$$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} -c_1 \\ -c_2 \end{bmatrix} \tag{20}$$

and numerical solution, e.g. as:

$$X = \frac{DET_X}{DET} \qquad Y = \frac{DET_Y}{DET} \tag{21}$$

and

$$DET_X = \begin{bmatrix} -c_1 & b_1 \\ -c_2 & b_2 \end{bmatrix} \qquad DET_Y = \begin{bmatrix} a_1 & -c_1 \\ a_2 & -c_2 \end{bmatrix} \qquad DET = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \tag{22}$$

However, there is always a problem if $DET \to 0$. The usual programmer's ***incorrect*** solution is:

IF $|DET| < eps$ **THEN** "singular case"; **EXIT**

Let us consider the equations above again and multiplying those equations by $w \neq 0$ we get:

$$a_1 wX + b_1 wY + c_1 w = 0 \qquad a_2 wX + b_2 wY + c_2 w = 0 \tag{23}$$

Now, the projective representation can be used and as $x = wX$ and $y = wY$, i.e.:

$$\begin{aligned} a_1 wX + b_1 wY + c_1 w = a_1 x + b_1 y + c_1 w = 0 \\ a_2 wX + b_2 wY + c_2 w = a_2 x + b_2 y + c_2 w = 0 \end{aligned} \tag{24}$$

in the vector notation then:

$$\boldsymbol{p}_1{}^T \boldsymbol{x} = 0 \qquad \boldsymbol{p}_2{}^T \boldsymbol{x} = 0 \tag{25}$$

where $\boldsymbol{x} = [x, y: w]^T$ is the intersection point in the homogeneous coordinates of two lines $\boldsymbol{p}_1 = [a_1, b_1 : c_1]^T$ and $\boldsymbol{p}_2 = [a_1, b_1 : c_1]^T$.

It is easy to show that the intersection point $\boldsymbol{x}$ expressed in the projective space can be computed as [11][13]:

$$\boldsymbol{x} = \boldsymbol{p_1} \times \boldsymbol{p_2} = \det \begin{bmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{bmatrix} = [x, y: w]^T \tag{26}$$

where $\boldsymbol{i} = [1,0: 0]^T$, $\boldsymbol{j} = [0,1: 0]^T$, $\boldsymbol{k} = [0,0: 1]^T$ are unit vectors in the projective space.

It is simple to prove that the above formula is correct. If two planes are parallel, then the coordinate $w = 0$, i.e. the intersection is in infinity.

The extended cross-product for E^4 has a form [17][18]:

$$\boldsymbol{q} = \boldsymbol{a} \times \boldsymbol{b} \times \boldsymbol{c} = \det \begin{bmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} & \boldsymbol{l} \\ a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \end{bmatrix} \tag{27}$$

where: $\boldsymbol{i} = [1,0,0,0]^T$, $\boldsymbol{j} = [0,1,0,0]^T$, $\boldsymbol{k} = [0,0,1,0]^T$, $\boldsymbol{l} = [0,0,0,1]^T$.

Now, due to the linearity it is possible to compute intersection of three planes $\boldsymbol{\rho}_1, \ldots, \boldsymbol{\rho}_3$ in P^3 as:

$$\boldsymbol{x} = \boldsymbol{\rho}_1 \times \boldsymbol{\rho}_2 \times \boldsymbol{\rho}_3 = \det \begin{bmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} & \boldsymbol{l} \\ a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{bmatrix} \tag{28}$$

where $\boldsymbol{\rho}_i = [a_i, b_i, c_i : d_i]^T$, i.e. $a_i X + b_i Y + c_i Z + d_i = 0$ and $\boldsymbol{x} = [x, y, z: w]^T$. It means that we can solve $\boldsymbol{Ax} = \boldsymbol{b}$ using the extended cross-product. Now, we use the principle of duality for solving $\boldsymbol{Ax} = \boldsymbol{0}$ case.

6 Duality

The projective representation offers also one very important property – principle of duality. The principle of duality in E^2 states that any theorem remains true when we interchange the words "point" and "line", "lie on" and "pass through", "join" and "intersection", "collinear" and "concurrent" and so on. Once the theorem has been established, the dual theorem is obtained as described above [1][5][7]. In other words, the principle of duality says that in all theorems it is possible to substitute the term "point" by the term "line" and the term "line" by the term "point" etc. in and the given theorem stays valid. Similar duality is valid for E^3 as well, i.e. the terms "point" and "plane" are dual etc. it can be shown that operations "join" and "meet" are dual as well.

7 Solution of Linear Systems

Let us consider two points in E^2 and a line passing those two points, i.e. the following linear system is to be solved:

$$aX_1 + bY_1 + c = 0 \qquad aX_2 + bY_2 + c = 0 \tag{29}$$

i.e. homogeneous linear system $\boldsymbol{Ax} = \boldsymbol{0}$ is to be solved:

$$\begin{bmatrix} X_1 & Y_1 & 1 \\ X_2 & Y_2 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{30}$$

It is actually one parametric set of solutions. If the equations are multiplied by $w_1, w_2 \neq 0$ and projective representation is used we can write:

$$\begin{bmatrix} x_1 & y_1 & w_1 \\ x_2 & y_2 & w_2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{31}$$

and solve it. It usually causes some problems in real implantation as programmers tend to set incorrectly some variable, e.g. $a = 1$ etc.

However, if we apply the principle of duality, i.e. lines dual to points and vice versa, then we can write:

$$\boldsymbol{p} = \boldsymbol{x_1} \times \boldsymbol{x_2} = \det \begin{bmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ x_1 & y_1 & w_1 \\ x_2 & y_2 & w_2 \end{bmatrix} = [a, b: c]^T \tag{32}$$

Similarly for the plane given by three points in E^3:

$$\boldsymbol{\rho} = \boldsymbol{x}_1 \times \boldsymbol{x}_2 \times \boldsymbol{x}_3 = \det \begin{bmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} & \boldsymbol{l} \\ x_1 & y_1 & z_1 & w_1 \\ x_2 & y_2 & z_2 & w_2 \\ x_3 & y_3 & z_3 & w_3 \end{bmatrix} = [a, b, c: d]^T \tag{33}$$

It means that a solution of a system of linear equations is projectively equivalent to the extended cross-product (outer product) application [17].

Generally, the system $\boldsymbol{Ax} = \boldsymbol{b}$ can be rewritten as:

$$[\boldsymbol{A}|-\boldsymbol{b}]\begin{bmatrix}\boldsymbol{x}\\ w\end{bmatrix} = \begin{bmatrix} a_{11} & \cdots & a_{1n} & -b_1 \\ \vdots & \ddots & \vdots & \vdots \\ a_{n1} & \cdots & a_{nn} & -b_n \end{bmatrix}\begin{bmatrix} x_1 \\ \vdots \\ x_n \\ w \end{bmatrix} = \begin{bmatrix}\boldsymbol{0}\\ 0\end{bmatrix} \tag{34}$$

and solution is given using the extended cross-product as:

$$\boldsymbol{\alpha}_1 \times \boldsymbol{\alpha}_2 \times \ldots \times \boldsymbol{\alpha}_n = [x_1, \ldots, x_n : w]^T \tag{35}$$

where $\boldsymbol{\alpha}_i = [a_{i1}, \ldots, a_{in} : b_i]$, $i = 1, \ldots, n$.

It should be noted, that the presented approach offers an unique approach to a solution of both types of the linear systems of equations, i.e. $\boldsymbol{Ax} = \boldsymbol{0}$ and $\boldsymbol{Ax} = \boldsymbol{b}$. It also offers possibility of further symbolic manipulations using standard vector operations, including dot product and cross-product.

Now, it is possible to apply the above presented concept with the Laplace transform to a solution of the linear system of differential equations.

8 Laplace Transform with Simple Examples

Let us consider again a simple system at Chap.2. of differential equations:

$$x' = 3x - 3y + 2 \qquad\qquad y' = -6x - t \tag{36}$$

with initial conditions $x(0) = 1, y(0) = -1$. Applying the Laplace transform, we obtain a system of linear algebraic equations with respect to x, y as:

$$\begin{aligned} sX(s) - x(0) &= 3X(s) - 3Y(s) + \frac{2}{s} \\ sY(s) - y(0) &= -6X(s) - \frac{1}{s^2} \end{aligned} \tag{37}$$

Including initial conditions this yield to:

$$\begin{aligned} (s-3)X(s) + 3Y(s) &= 1 + \frac{2}{s} \\ 6X(s) + sY(s) &= -1 - \frac{1}{s^2} \end{aligned} \tag{38}$$

It means that the system described by equations:

$$\begin{bmatrix} s-3 & 3 \\ 6 & s \end{bmatrix}\begin{bmatrix} X(s) \\ Y(s) \end{bmatrix} = \begin{bmatrix} \dfrac{s+2}{s} \\ -\dfrac{s^2+1}{s^2} \end{bmatrix} \tag{39}$$

In the projective representation, it is represented as:

$$\bar{\boldsymbol{x}}(s) = \boldsymbol{\xi}_1 \times \boldsymbol{\xi}_2 = \det\begin{bmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ s-3 & 3 & -\dfrac{s+2}{s} \\ 6 & s & \dfrac{s^2+1}{s^2} \end{bmatrix} = [\bar{x}(s), \bar{y}(s) : \bar{w}(s)]^{\mathrm{T}} \tag{40}$$

where:

$$\boldsymbol{\xi}_1 = \left[s-3, 3: -\frac{s+2}{s}\right]^T \qquad \boldsymbol{\xi}_2 = \left[6, s: \frac{s^2+1}{s^2}\right]^T \tag{41}$$

Applying the extended cross-product, a solution is obtained:

$$\overline{\boldsymbol{x}}(s) = [\bar{x}(s), \bar{y}(s): \bar{w}(s)]^{\mathrm{T}} = \begin{bmatrix} 3\dfrac{s^2+1}{s^2} + \dfrac{s+2}{s}s \\ -6\dfrac{s+2}{s} - (s-3)\dfrac{s^2+1}{s^2} \\ s(s-3) - 18 \end{bmatrix} \tag{42}$$

i.e.

$$\begin{gathered} \bar{x}(s) = 3\frac{s^2+1}{s^2} + s + 2 \\ \bar{y}(s) = -6\frac{s+2}{s} - (s-3)\frac{s^2+1}{s^2} \\ \bar{w}(s) = s(s-3) - 18 \end{gathered} \tag{43}$$

If the conversion to the Euclidean space representation is needed, then:

$$\begin{aligned} X(s) = \frac{\bar{x}(s)}{\bar{w}(s)} = \frac{3\dfrac{s^2+1}{s^2} + s + 2}{s(s-3) - 18} = \frac{s^2(s+2) + 3s^2 + 3}{s^2(s^2 - 3s - 18)} \\ = \frac{s^3 + 5s^2 + 3}{s^2(s^2 - 3s - 18)} \end{aligned} \tag{44}$$

and

$$Y(s) = \frac{\bar{y}(s)}{\bar{w}(s)} = -\frac{(s-3)\dfrac{s^2+1}{s^2} + 6\dfrac{s+2}{s}}{s(s-3) - 18} = -\frac{s^3 + 3s^2 + 13s - 3}{s^2(s^2 - 3s - 18)} \tag{45}$$

Of course, the above presented approach can be applied with including general (unspecified) initial conditions.

Let us consider another example with unspecified initial conditions (problem formulation taken from [22]):

$$x'(t) = y(t) \qquad y'(t) = -4x(t) + \sin(\omega t) \tag{46}$$

Using the Laplace transform and projective representation we obtain:

$$sX(s) - x(0) = Y(s) \qquad sY(s) - y(0) = -4X(s) + \frac{\omega}{s^2 + \omega^2} \tag{47}$$

and in the matrix form:

$$\begin{bmatrix} s & -1 & -x(0) \\ 4 & s & -y(0) - \dfrac{\omega}{s^2 + \omega^2} \end{bmatrix} \begin{bmatrix} \bar{x} \\ \bar{y} \\ \bar{w} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{48}$$

The solution is then given as:

$$\bar{\boldsymbol{x}}(s) = \boldsymbol{\xi}_1 \times \boldsymbol{\xi}_2 = \det \begin{bmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ s & -1 & -x(0) \\ 4 & s & -y(0) - \dfrac{\omega}{s^2+\omega^2} \end{bmatrix} \tag{49}$$

$$\boldsymbol{\xi}_1 = [s, -1 : -x(0)]^T \qquad \boldsymbol{\xi}_2 = \left[4, s : -y(0) - \frac{\omega}{s^2+\omega^2}\right]^T \tag{50}$$

Therefore

$$\bar{\boldsymbol{x}}(s) = [\bar{x}(s), \bar{y}(s) : \bar{w}(s)]^{\mathrm{T}} = \begin{bmatrix} \dfrac{\omega}{s^2+\omega^2} + y(0) + sx(0) \\ s\left(y(0) + \dfrac{\omega}{s^2+\omega^2}\right) + 4x(0) \\ s^2 + 4 \end{bmatrix} \tag{51}$$

i.e

$$\begin{gathered} \bar{x}(s) = \frac{\omega}{s^2+\omega^2} + y(0) + sx(0) \\ \bar{y}(s) = s\left(y(0) + \frac{\omega}{s^2+\omega^2}\right) - 4x(0) \\ \bar{w}(s) = s^2 + 4 \end{gathered} \tag{52}$$

If the conversion to the Euclidean space representation is needed, then:

$$\begin{aligned} X(s) &= \frac{\bar{x}(s)}{\bar{w}(s)} = \frac{\dfrac{\omega}{s^2+\omega^2} + y(0) + sx(0)}{s^2+4} \\ Y(s) &= \frac{\bar{y}(s)}{\bar{w}(s)} = -\frac{s\left(y(0) + \dfrac{\omega}{s^2+\omega^2}\right) + 4x(0)}{s^2+4} \end{aligned} \tag{53}$$

Now, the inverse Laplace transform is to be used or splitting to partial fractions using known patterns to obtain solution in the space-time domain [25].

The presented approach demonstrates an equivalence of system of linear equations and the extended cross product, a more general approach can be found in [6][8][19][23].

9 Symbolic Manipulations and Transformations

Let us consider simple examples in E^2, resp. P^2, for simplicity of explanation. Let $\boldsymbol{M}$ is a regular transformation matrix $n \times n$, where $n = 2$, applied to vectors $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2$:

$$\bar{\boldsymbol{x}}(s) = [\bar{x}(s), \bar{y}(s) : \bar{w}(s)]^{\mathrm{T}} = \boldsymbol{\xi}_1 \times \boldsymbol{\xi}_2 \tag{54}$$

We want to know, what will be the result after that transformation, i.e.

$$\bar{\boldsymbol{x}}_1(s) = [\bar{x}_1(s), \bar{y}_1(s) : \bar{w}_1(s)]^{\mathrm{T}} = (\boldsymbol{M}\boldsymbol{\xi}_1) \times (\boldsymbol{M}\boldsymbol{\xi}_2) \tag{55}$$

It can be shown, that in the case of the E^3 case, the following identity is valid:

$$(\boldsymbol{M}\boldsymbol{a}) \times (\boldsymbol{M}\boldsymbol{b}) = \det(\boldsymbol{M})\boldsymbol{M}^{-T}(\boldsymbol{a} \times \boldsymbol{b}) \tag{56}$$

However, as the representation of $\overline{\boldsymbol{x}}(s)$ is in the projective space P^2, we can write:

$$(\boldsymbol{M}\boldsymbol{\xi}_1) \times (\boldsymbol{M}\boldsymbol{\xi}_2) = \det(\boldsymbol{M})\boldsymbol{M}^{-T}(\boldsymbol{\xi}_1 \times \boldsymbol{\xi}_2) \triangleq \boldsymbol{M}^{-T}(\boldsymbol{\xi}_1 \times \boldsymbol{\xi}_2) \tag{57}$$

where: $\triangleq$ means "projectively" equivalent.
It means that:

$$\overline{\boldsymbol{x}}_1(s) = [\bar{x}_1(s), \bar{y}_1(s) : \bar{w}_1(s)]^{\mathrm{T}} = (\boldsymbol{M}\boldsymbol{\xi}_1) \times (\boldsymbol{M}\boldsymbol{\xi}_2) \triangleq \boldsymbol{M}^{-T}\overline{\boldsymbol{x}}(s) \tag{58}$$

It should be noted that similarly for the k-dimensional case can be written:

$$(\boldsymbol{M}\boldsymbol{a}_1) \times \ldots \times (\boldsymbol{M}\boldsymbol{a}_k) = [\det(\boldsymbol{M})]^{k-1}\boldsymbol{M}^{-T}(\boldsymbol{a}_1 \times \ldots \times \boldsymbol{a}_k) \tag{59}$$

and therefore also

$$\overline{\boldsymbol{x}}_1(s) = [\bar{x}_1(s), \bar{y}_1(s) : \bar{w}_1(s)]^{\mathrm{T}} = (\boldsymbol{M}\boldsymbol{\xi}_1) \times \ldots \times (\boldsymbol{M}\boldsymbol{\xi}_2) \triangleq \boldsymbol{M}^{-T}\overline{\boldsymbol{x}}(s) \tag{60}$$

as $\overline{\boldsymbol{x}}(s)$ is represented by homogeneous coordinates in the projective space.

This is a very important findings as it says that if the original vector $\overline{\boldsymbol{x}}(s)$ in the n-dimensional space is transformed by a transformation $\boldsymbol{M}$, then the result can be easily determined. The transformation $\boldsymbol{M} = \boldsymbol{M}(s)$, in general

10 Conclusion

A new approach to solution of a system of differential equations is shortly described in this contribution. It was shown that a solution of linear system of equations is equivalent to the extended cross-product using projective space representation. There are the following significant results:

- division operation is not needed if the final result can remain in the projective representation
- precision of numerical computation significantly higher as both mantissa length of x, w and both exponents ranges are used
- as the result is represented as extended cross-product vector-vector symbolic operations can be applied instead of numerical solution
- the result of the Laplace transformation is represented as generalized cross-product and can be used for symbolic manipulation using vector-vector or matrix-vector operation as well

The presented approach is applicable to n-dimensional system of algebraic linear equations. It is expected, that the presented approach will be applicable also in other engineering fields and research areas as well. It should be noted that the presented approach is directly applicable to multi-dimensional dynamical systems.

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