Smolik,M., Skala,V.: Vector Field Interpolation with Radial Basis Functions
SIGRAD 2016, pp.15-21, Sweden, 2016

SIGRAD 2016
M. Hayashi (Editors)

Vector Field Interpolation with Radial Basis Functions

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Abstract

This paper presents a new approach for the Radial Basis Function (RBF) interpolation of a vector field. Standard approaches for interpolation randomly select points for interpolation. Our approach uses the knowledge of vector field topology and selects points for interpolation according to the critical points location. We presents the results of interpolation errors on a vector field generated from an analytical function.

Categories and Subject Descriptors (according to ACM CCS): G.1.1 [Numerical analysis]: Interpolation—Interpolation formulas

1. Introduction

Interpolation is probably the most frequent operation used in computational methods. Several methods have been developed for data interpolation, but they expect some kind of data "ordering", e.g. structured mesh, rectangular mesh, unstructured mesh, etc. However, in many engineering problems, data are not ordered and they are scattered in $d-$dimensional space, in general. Usually, in technical applications, the scattered data are tessellated using triangulation but this approach is quite prohibitive for the case of $d-$dimensional data interpolation because of the computational cost.

Interpolating scattered vector data on a surface becomes frequent in applied problem solutions. There are applications for vector field decomposition [EJF09], for vector field design system for surfaces that allows the user to control the number of singularities in the vector field and their placement [ZMT06]. [MZT*14] uses the vector field interpolation for estimating robust point correspondences between two sets of points.

2. Vector Field

Vector fields on surfaces are important objects, which appear frequently in scientific simulation in CFD (Computational Fluid Dynamics) or modeling by FEM (Finite Element Method). To be visualized, such vector fields are usually linearly approximated for the sake of simplicity and performance considerations.

The vector field can be easily analyzed when having an approximation of the vector field near some location point. The important places to be analyzed are so called critical points. Analyzing the vector field behavior near these points gives us the information about the characteristic of the vector field.

2.1. Critical Point

Critical points $\boldsymbol{x}_0$ of the vector field are points at which the magnitude of the vector vanishes

$$\frac{d\boldsymbol{x}}{dt} = \boldsymbol{v}(\boldsymbol{x}) = \boldsymbol{0}, \tag{1}$$

i.e. all components are equal to zero

$$\begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{2}$$

A critical point is said to be isolated, or simple, if the vector field is non vanishing in an open neighborhood around the critical point. Thus for all surrounding points $\boldsymbol{x}_\varepsilon$ of the critical point $\boldsymbol{x}_0$ the equation (1) does not apply, i.e.

$$\frac{d\boldsymbol{x}_\varepsilon}{dt} \neq \boldsymbol{0}, \tag{3}$$

At critical points, the direction of the field line is indeterminate, and they are the only points in the vector field were field lines can intersect (asymptotically). The terms singular point, null point, neutral point or equilibrium point are also frequently used to describe critical points.

These points are important because together with the

nearby surrounding vectors, they have more information encoded in them than any such group in the vector field, regarding the total behavior of the field.

2.2. Linearization of Vector Field

Critical points can be characterized according to the behavior of nearby tangent curves. We can use a particular set of these curves to define a skeleton that characterizes the global behavior of all other tangent curves in the vector field. An important feature of differential equations is that it is often possible to determine the local stability of a critical point by approximating the system by a linear system. These approximations are aimed at studying the local behavior of a system, where the nonlinear effects are expected to be small. To locally approximate a system, the Taylor series expansion must be utilized locally to find the relation between $\boldsymbol{v}$ and position $\boldsymbol{x}$, supposing the flow $\boldsymbol{v}$ to be sufficiently smooth and differentiable. In such case, the expansion of $\boldsymbol{v}$ around the critical points $\boldsymbol{x}_0$ is

$$\boldsymbol{v}(\boldsymbol{x}) = \boldsymbol{v}(\boldsymbol{x}_0) + \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{x}}(\boldsymbol{x} - \boldsymbol{x}_0). \tag{4}$$

As $\boldsymbol{v}(\boldsymbol{x}_0)$ is according to (1) equal zero for critical points, we can rewrite equation (4) using matrix notation

$$\begin{bmatrix} v_x \\ v_y \end{bmatrix} = \begin{bmatrix} \frac{\partial v_x}{\partial x} & \frac{\partial v_x}{\partial y} \\ \frac{\partial v_y}{\partial x} & \frac{\partial v_y}{\partial y} \end{bmatrix} \begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix} \tag{5}$$

$$\boldsymbol{v} = \boldsymbol{J} \cdot (\boldsymbol{x} - \boldsymbol{x}_0), \tag{6}$$

where $\boldsymbol{J}$ is called Jacobian matrix and characterizes the vector field behavior around a critical point $\boldsymbol{x}_0$.

2.3. Classification of Critical Points

There exist a finite set of fundamentally different critical points, defined by the number of inflow and outflow directions, spiraling structures etc., and combinations of these. Since the set is finite, each critical point can be classified. Such a classification defines the field completely in a close neighborhood around the critical point. By knowing the location and classification of critical points in a vector field, the topology of the field is known in small areas around these. Assuming a smooth transition between these areas, one can construct a simplified model of the whole vector field. Such a simplified representation is useful, for instance, in compressing vector field data into simpler building blocks [PS97].

The critical points are classified based on the vector field around these points. The information derived from the classification of critical points aids the information selection process when it comes to visualizing the field. By choosing seed points for field lines based on the topology of critical points, field lines encoding important information is ensured.

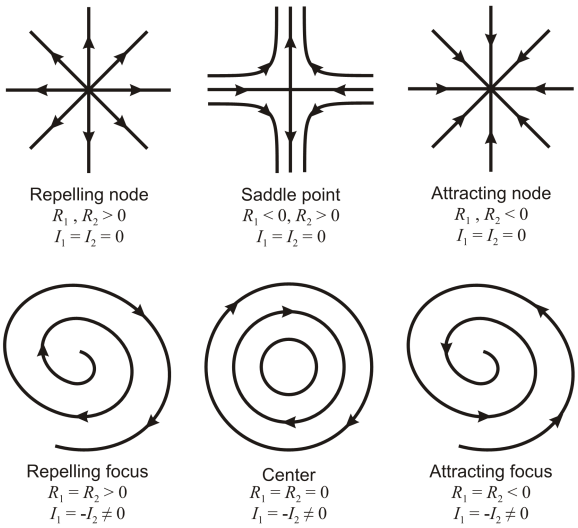


Figure 1: Classification of 2*D* first order critical points. R_1, R_2 denote the real parts of the eigenvalues of the Jacobian matrix while I_1, I_2 denote their imaginary parts (from [HH89]).

A more advanced approach is to connect critical points, and use the connecting lines and surfaces to separate areas of different flow topology [HH89], [WTS*05].

The fact that a linear model can be used to study the behavior of a nonlinear system near a critical point is a powerful one [HH89]. We can use the Jacobian matrix to characterize the vector field and the behavior of nearby tangent curves, for nondegenerate critical point.

The eigenvalues and the eigenvectors of Jacobian matrix are very important for vector field classification and description, see Figure 1. A real eigenvector of the Jacobian matrix defines a direction such that if we move slightly from the critical point in that direction, the field is parallel to the direction we moved. Thus, at the critical point, the real eigenvectors are tangent to the trajectories that end on the point. The sign of the corresponding eigenvalue determines whether the trajectory is outgoing (repelling) or incoming (attracting) at the critical point. The imaginary part of an eigenvalue denotes circulation about the point.

3. Radial Basis Functions

The Radial basis functions (RBF) is a technique for scattered data interpolation [PS11] and approximation [Fas07], [Ska15]. The RBF interpolation and approximation is computationally more expensive, because input data are not ordered and there is no known relation between them. Although the RBF has higher computational cost, it can be used for d-dimensional problem solution in many applications, e.g. solution of partial differential equations, image re-

construction, neural networks, fuzzy systems, GIS systems, optics etc.

The RBF is a function whose value depends only on the distance from some center point. Due to the use of the distance functions, the RBFs can be easily implemented to reconstruct the surface using scattered data in 2D, 3D or higher dimensional spaces. It should be noted that the RBF interpolation is not separable.

Radial function interpolants have a nice property of being invariant under all Euclidean transformations, i.e. translations, rotations and reflections. It means that it does not matter whether we first compute the RBF interpolation function and then apply a Euclidean transformation, or if we first transform all the data and then compute the radial function interpolants. This is result of the fact that Euclidean transformations are characterized by orthogonal transformation matrices and are therefore 2 norm invariant. Radial basis functions can be divided into two groups according to their influence. First group are "global" RBF [Sch79], for example:

$$\begin{aligned}
&\text{Thin Plate Spline (TPS)} && \varphi(r) = r^2 \log r \\
&\text{Gauss function} && \varphi(r) = e^{-(\varepsilon r)^2} \\
&\text{Inverse Quadric (IQ)} && \varphi(r) = \frac{1}{1+(\varepsilon r)^2} \\
&\text{Inverse Multiquadric (IMQ)} && \varphi(r) = \frac{1}{\sqrt{1+(\varepsilon r)^2}} \\
&\text{Multiquadric (MQ)} && \varphi(r) = \sqrt{1+(\varepsilon r)^2}
\end{aligned} \tag{7}$$

where ε is the shape parameter of radial basis function [FP08].

The "local" RBF were introduced by [Wen06] as Compactly Supported RBF (CSRBF) and satisfy the following condition

$$\varphi(r) = (1-r)_+^q P(r) = \begin{cases} (1-r)^q P(r) & 0 \le r \le 1 \\ 0 & r > 1 \end{cases} \tag{8}$$

where $P(r)$ is a polynomial function and q is a parameter. Typical examples of CSRBF are

$$\begin{aligned}
\varphi_1(r) &= (1-\varepsilon r)_+ \\
\varphi_2(r) &= (1-\varepsilon r)_+^3 (3\varepsilon r+1) \\
\varphi_3(r) &= (1-\varepsilon r)_+^5 (8(\varepsilon r)^2 + 5\varepsilon r + 1) \\
\varphi_4(r) &= (1-\varepsilon r)_+^2 \\
\varphi_5(r) &= (1-\varepsilon r)_+^3 (4\varepsilon r+1) \\
\varphi_6(r) &= (1-\varepsilon r)_+^6 (35(\varepsilon r)^2 + 18\varepsilon r + 3) \\
\varphi_7(r) &= (1-\varepsilon r)_+^8 (32(\varepsilon r)^3 + 25(\varepsilon r)^2 + 8\varepsilon r + 1) \\
\varphi_8(r) &= (1-\varepsilon r)_+^3 \\
\varphi_9(r) &= (1-\varepsilon r)_+^3 (5\varepsilon r+1) \\
\varphi_{10}(r) &= (1-\varepsilon r)_+^7 (16(\varepsilon r)^2 + 7\varepsilon r + 1)
\end{aligned} \tag{9}$$

where ε is the shape parameter of radial basis function, see Figure 2 for visualization of (9).

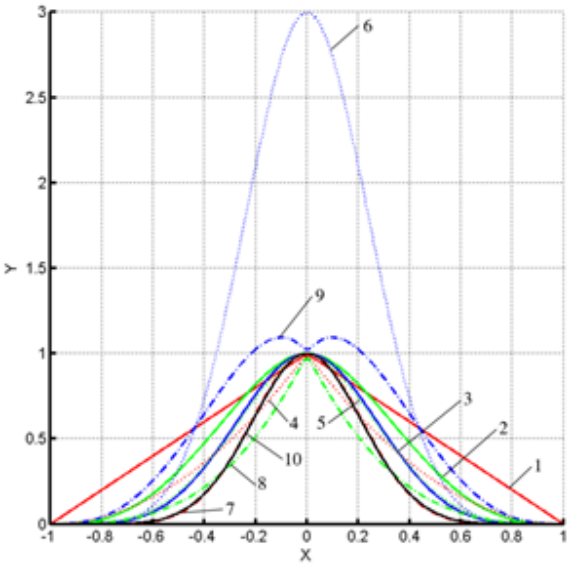


Figure 2: Examples of CSRBF (from [US04])

3.1. Radial Basis Function Interpolation

The RBF interpolation was originally introduced by [Har71] and is based on computing the distance of two points in the k-dimensional space and is defined by a function

$$f(\boldsymbol{x}) = \sum_{j=1}^{M} \lambda_j \varphi(\|\boldsymbol{x} - \boldsymbol{x}_j\|) \tag{10}$$

where λ_j are weights of the RBFs, M is the number of the radial basis functions, i.e. the number of interpolation points, and φ is the radial basis function. For a given dataset of points with associated values, i.e. in the case of scalar values $\{\boldsymbol{x}_i, h_i\}_1^M$, the following linear system of equations is obtained

$$h_i = f(\boldsymbol{x}_i) = \sum_{j=1}^{M} \lambda_j \varphi(\|\boldsymbol{x}_i - \boldsymbol{x}_j\|) \quad \text{for } \forall i \in \{1,\dots,M\} \tag{11}$$

where λ_j are weights to be computed, see Figure 3 for visual interpretation of (10) or (11) for a $2\frac{1}{2}D$ function.

Equation (11) can be rewritten in a matrix form as

$$\boldsymbol{A}\boldsymbol{\lambda} = \boldsymbol{h} \tag{12}$$

where matrix $\boldsymbol{A}$ is symmetrical, as $\|\boldsymbol{x}_i - \boldsymbol{x}_j\| = \|\boldsymbol{x}_j - \boldsymbol{x}_i\|$.

The RBF interpolation can be done using "global" or "local" functions. When using "global" radial basis functions the matrix $\boldsymbol{A}$ will be full, but when using "local" radial basis

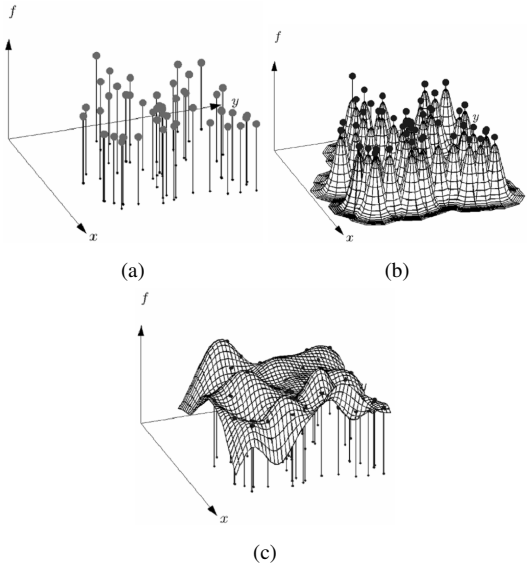


Figure 3: Data values $\{\boldsymbol{x}_i, h_i\}_1^M$ (Figure 3a), the RBF collocation functions (Figure 3b), the resulting interpolant (Figure 3c). (From [FW09]).

functions the matrix $\boldsymbol{A}$ will be sparse, which can be beneficial when solving the system of linear equations $\boldsymbol{A\lambda} = \boldsymbol{h}$.

In the case of the vector data, i.e. $\{\boldsymbol{x}_i, \boldsymbol{h}_i\}_1^M$ values $\boldsymbol{h}_i$ are actually vectors, the RBF is to be performed for each coordinate of $\boldsymbol{h}_i$.

4. Vector Field RBF Approximation

Vector fields are results of numerical simulations or data measuring process. This kind of vector field data has discrete representation, but an analytical formula describing the vector filed is much more useful. We will show how to approximate a vector field using radial basis functions.

A very important future of a vector field are its critical points. The interpolation must preserve positions and types of all critical points. Thus, the RBF interpolation should interpolate the vector field at all positions of critical points to preserve their positions. To preserve their types, we should include few more points in the neighborhood of each critical point to the interpolation. The number of points in the neighborhood was experimentally chosen to be 4, as more points does not improve the interpolation in any significant way. Points in the neighborhood of a critical point $\boldsymbol{x}_0 = [x_0, y_0]^T$ are chosen using the following formula

$$\begin{bmatrix} P_x^{(k)} \\ P_y^{(k)} \end{bmatrix} = \begin{bmatrix} x_0 + r\sin(k\frac{\pi}{2}) \\ y_0 + r\cos(k\frac{\pi}{2}) \end{bmatrix}. \tag{13}$$

where $k \in \{0,1,2,3\}$ and r is a small number depending on the distance of critical points, where the distance to the nearest critical point should be $\gg r$.

This set of critical points together with their neighborhood points can be interpolated using RBF (11), note that each component of vectors $\boldsymbol{v} = [v_x, v_y]^T$ is interpolated separately. This interpolation will preserve the location of critical points together with their types.

To get more accurate interpolation formula of a vector field at points $\boldsymbol{x} \in [x_{min}, x_{max}] \times [y_{min}, y_{max}]$ we can include some more random points from this interval into the interpolation. The improvement of quality depending on the number of additionally included points will be shown in the following chapter.

5. Results

The results will be demonstrated on an analytical vector field, as we can measure the interpolation errors precisely. The analytical vector field, that we choose as an example, is described with the following equation

$$\begin{bmatrix} v_x \\ v_y \end{bmatrix} = \begin{bmatrix} x(\frac{1}{2}x^2 + \frac{1}{2}) + y(-x + (\frac{1}{2}y - 1)y + \frac{1}{2}) \\ \frac{1}{2}x^2 y + x(-\frac{1}{2}y^2 + y - \frac{1}{2}) + \frac{1}{2}y - 1 \end{bmatrix} \tag{14}$$

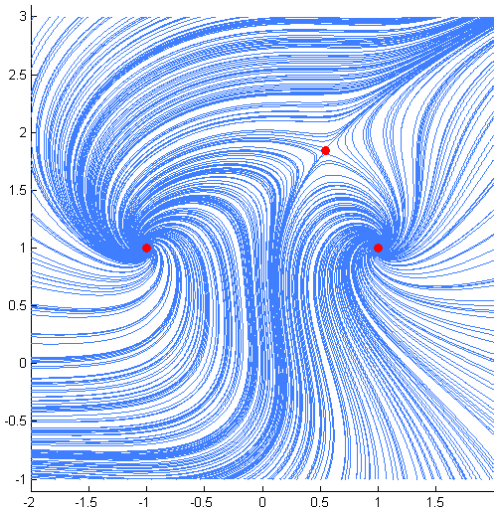
this vector field (14) has three critical points $\boldsymbol{x}_0$

$$\begin{aligned} &\text{source location:} && \boldsymbol{x}_0 = [-1,1]^T \\ &\text{source location:} && \boldsymbol{x}_0 = [1,1]^T \\ &\text{saddle location:} && \boldsymbol{x}_0 = [0.543689, 1.83929]^T. \end{aligned} \tag{15}$$

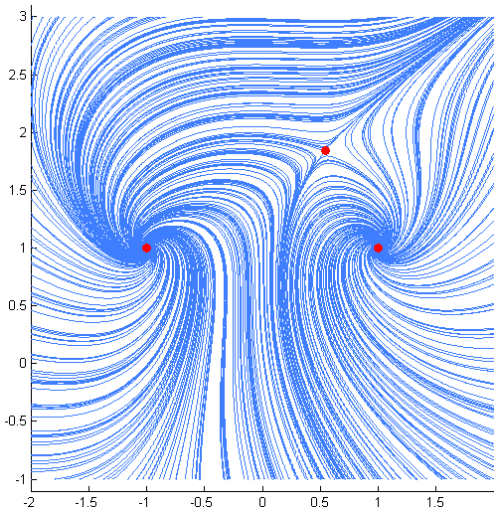
The vector field (14) will be interpolated and tested on interval $[-2,2] \times [-1,3]$, as all important features will be visible. The RBF function used for interpolation is a Gauss radial basis function and the shape parameter ε was experimentally selected as $\varepsilon = 1$.

Vector field (14) can be interpolated using 3 critical point positions and 12 more neighborhood points, i.e. 4 neighborhood points for each critical point. The neighborhood points are computed with (13) and the parameter $r = 0.1$. The v_x component of the vector field is interpolated with one RBF and the v_y component of the vector field is interpolated with one RBF as well. The phase portrait of original analytical vector field (14) is visualized in Figure 4a and the phase portrait of RBF interpolated vector field is visualized in Figure 4b. It can be seen, that both phase portraits look very similar and have the same vector field topology. Moreover, the critical points location is identical, as the average length of displacement error for all critical points is $7.0283 \cdot 10^{-8}$, which is only a numerical error of the critical points location algorithm.

We computed the interpolation error for v_x and v_y and visualized it in Figure 5. It can be seen that the interpolation error is getting higher as the distance from critical

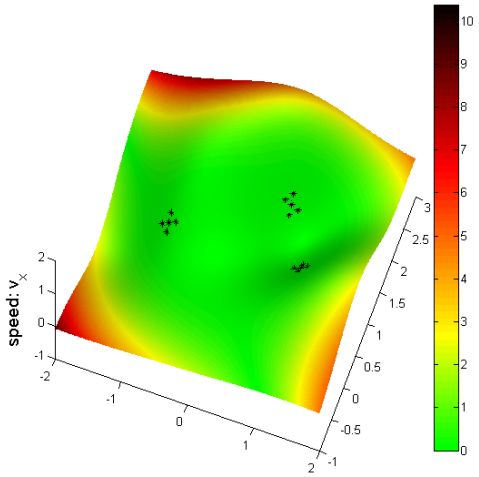


(a)

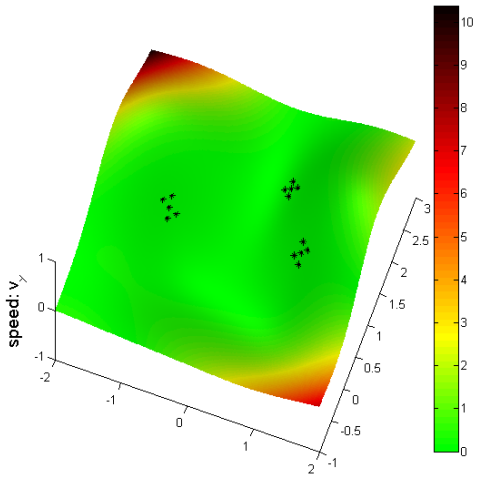


(b)

Figure 4: Phase portrait of the vector field (14) (Figure 4a) and phase portrait of a RBF interpolation using only 15 reference points (3 critical points plus three times 4 neighborhood points) (Figure 4b).



(a)



(b)

Figure 5: Interpolation error of RBF interpolation using only 15 reference points (3 critical points plus three times 4 neighborhood points). Interpolation error of v_x (Figure 5a) and interpolation error of v_y (Figure 5b).

points increases. The average error of vector length at interval $[-2,2] \times [-1,3]$ is 1.7943 (the vector length varies from 0 to 12.6194) and the average error of vector angular displacement is 0.1966 $[rad]$.

The vector field (14) was interpolated using 3 critical points locations plus three times 4 neighborhood points. We can include few more randomly distributed points into the interpolation to reduce the distance error from (14). We choose to generate additional 85 points from interval $[-2,2] \times [-1,3]$, so the interpolation of vector field will contain 10^2 points in total. This interpolation of vector field is visualized in a phase portrait, see Figure 6 and Figure 4a for comparison with original phase portrait.

We computed the interpolation error for v_x and v_y and visualized it in Figure 7. It can be seen that the interpolation error is close to zero except for locations on the border. The average error of vector length at interval $[-2,2] \times [-1,3]$ is 0.0549 (note that the vector length varies from 0 to 12.6194) and the average error of vector angular displacement is 0.0065 $[rad]$.

The average vector length error and the average vector angular displacement error were measured for different number of interpolated points. A number of points k is used as

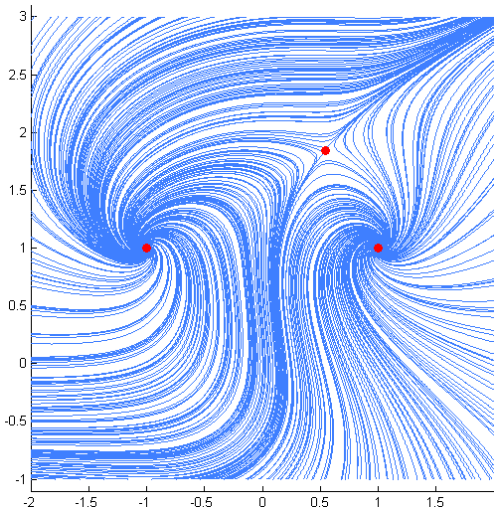


Figure 6: Phase portrait of a vector field RBF interpolation of (14) using 100 reference points (3 critical points plus three times 4 neighborhood points plus 85 randomly distributed points).

added points for the RBF interpolation, thus the RBF interpolation uses $(k+3+3\cdot 4)$ points for interpolation of vector field, i.e. k randomly distributed points from interval $[-2,2]\times[-1,3]$ plus 3 critical points plus three times 4 neighborhood points. Number k was tested from 0 to 400 fifty times for each k with step $\Delta k = 1$ and results are visualized in Figure 8.

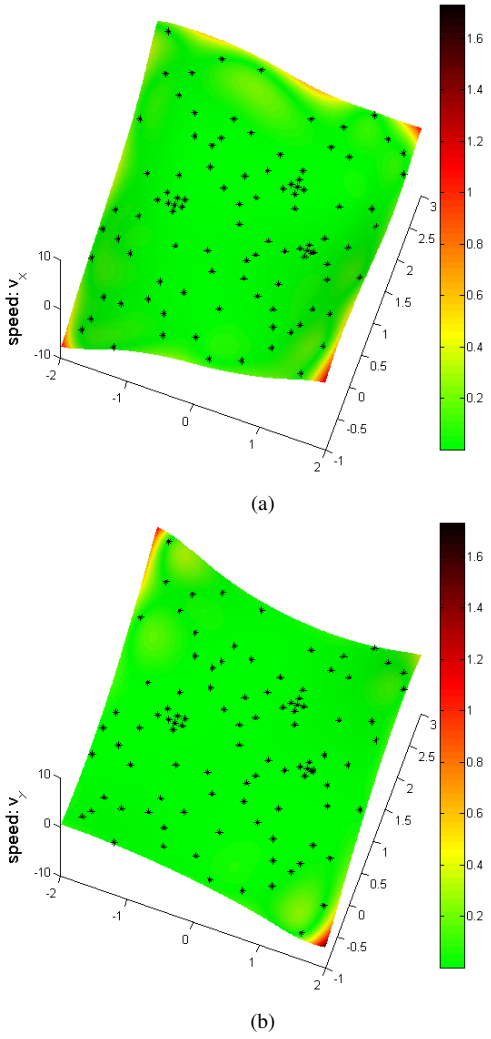
It can be seen that both errors in Figure 8 decrease with increasing number k of added points for the interpolation of vector field. According to the required accuracy of the interpolation, the user can select the minimal necessary number of added points and perform the interpolation according to the algorithm proposed.

6. Conclusions

We presented a new and easy to implement approach for the vector field approximation using radial basis functions. In general, it can be used in any $d-$dimensional space, although the results were presented only for $2D$ vector field. The proposed RBF interpolation proved the ability to approximate a vector field when preserving the location of critical points and the vector field topology as well.

The proposed approach offers not only analytical description of the discrete data of vector field, but also a significant data compression. This might be a significant feature for "progressive vector field visualization" approach.

In future, the proposed approach will be deeply explored for $t-$varying data sets together with other aspects for very large vector field data set interpolation. The more sophisticated placement of interpolation points around critical points will be deeply explored as well.



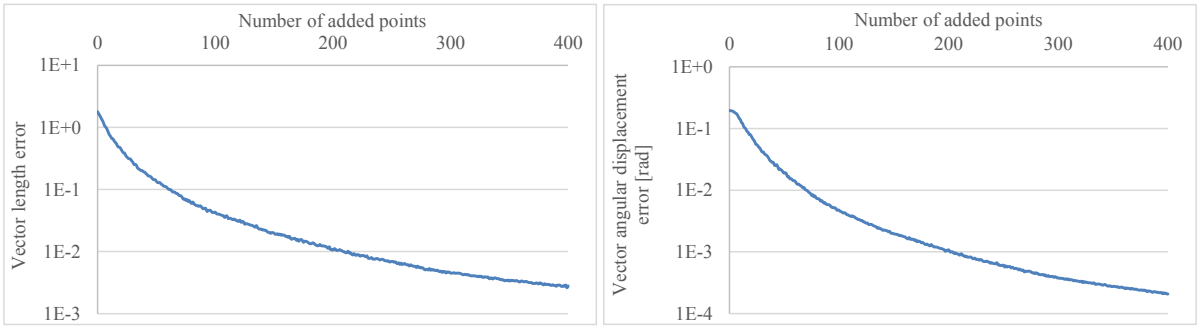
(a)

(b)

Figure 7: Interpolation error of RBF interpolation using 100 reference points (3 critical points plus three times 4 neighborhood points plus 85 randomly distributed points). Interpolation error of v_x (Figure 7a) and interpolation error of v_y (Figure 7b).

Acknowledgement

The authors would like to thank their colleagues at the University of West Bohemia, Plzen, for their discussions and suggestions, and anonymous reviewers for their valuable comments and hints provided. The research was supported by MSMT CR project LH12181 and SGS 2016-013.



(a) (b)

Figure 8: Average errors of the RBF interpolation of vector field (14) using k added reference points, i.e. 3 critical points plus three times 4 neighborhood points plus k randomly distributed points, where $k \in \{0,\dots,400\}$. The vector field length error, note that the vector length varies from 0 to 12.6194 (Figure 8a) and the vector field angular displacement error (Figure 8b).

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