Geometric Transformations and Duality for Virtual Reality and Haptic Systems

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Abstract. Virtual reality and haptic systems use geometric transformations with points represented in homogeneous coordinates extensively. In many cases interpolation and barycentric coordinates are used. However, developers do not fully use properties of projective representation to make algorithms stable, robust and faster. This paper describes geometric transformations and principle of duality which enables to solve some problems effectively.

Keywords: projective space, homogeneous coordinates, principle of duality, barycentric coordinates, linear system of equations, outer product, cross product GPU computation

1 Introduction

Geometric transformations with rigid objects are used in computer graphics, virtual reality and haptic systems. Homogeneous coordinates are mostly introduced with regard to geometric transformations only. However, if projective extension of the Euclidean space, which uses homogeneous coordinates natively, is used for reformulation of algorithms, not necessarily geometrical only, simple, faster and more robust algorithms are obtained. Also if principle of duality is used, users can obtain novel algorithms with better properties.

In this paper some principles are described and demonstrated on simple geometrical problems. Significant advantage of projective algorithms reformulation is that naturally supports vector-vector architectures like SSE and GPU.

2 Projective Representation and Principle of Duality

In geometry the Euclidean coordinates are used in general, as the Euclidean space has a “metric”, i.e. a distance of two points in E^2 is computed as:

$$d = \sqrt{(\Delta X)^2 + (\Delta Y)^2} \qquad (1)$$

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A point $\boldsymbol{X}$ in E^2 is represented as $\boldsymbol{X} = (X, Y) \in E^2$. However in geometry lines and planes are used as well and they can be described in an implicit form:

$$F(\boldsymbol{X}) = 0 \quad (2)$$

This equation can be multiplied by any constant $q \neq 0$ and the given geometric primitive does not change. It means that a one parametric formulation is given.

Projective Extension of the Euclidean Space

The projective extension on the Euclidean space is simple and it is defined by:

$$X = {}^{x}/_{w} \qquad Y = {}^{y}/_{w} \qquad w \neq 0 \quad (3)$$

Coordinates in projective representation are given by homogeneous coordinates [1], [10]. The point $\boldsymbol{X} = (X, Y) \in E^2$ is represented by a vector $\boldsymbol{x} = [x, y{:}\, w]^T \in P^2$. In many cases, especially in mathematically oriented texts, notification $[a_0{:}\, a_1, \dots, a_n,]$ is used, where homogeneous coordinate $w \equiv a_0$. This is more convenient notation especially for n-dimensional applications. Homogeneous coordinates are mostly used for a point representation. However in geometric algorithms lines and planes are also important primitives.

A line in E^2 is defined as:

$$aX + bY + d = 0 \quad (4)$$

If multiplied by $w \neq 0$, then:

$$awX + bwY + dw = 0 \quad (5)$$

and therefore a line can be defined, if homogeneous coordinates are used, as:

$$ax + by + dw = \boldsymbol{a} \cdot \boldsymbol{x} = \boldsymbol{a}^T\boldsymbol{x} = 0 \quad (6)$$

where: $\boldsymbol{a} = [a, b{:}\, d]^T$ and $\boldsymbol{x} = [x, y{:}\, w]^T$. It can be seen that the formula is more "compact".

In the case of E^3 a point is given as $\boldsymbol{X} = (X, Y, Z) \in E^3$ in the Euclidean space or as $\boldsymbol{x} = [x, y, z{:}\, w]^T \in P^3$ in the projective space. A plane is given as:

$$ax + by + cz + dw = \boldsymbol{a} \cdot \boldsymbol{x} = \boldsymbol{a}^T\boldsymbol{x} = 0 \quad (7)$$

where: $\boldsymbol{a} = [a, b, c{:}\, d]^T$ and $\boldsymbol{x} = [x, y, z{:}\, w]^T$.

It can be seen that homogeneous representation can be used also for a fraction a/b representation in order to postpone a division operation and obtain higher precision of computation as two float representations in the fraction are used and division operation is postponed.

Principle of Duality

Principle of duality is one of the most important principles used in geometry. From the line or plane equation $\boldsymbol{a}^T\boldsymbol{x} = 0$ can be seen that the meaning of symbols $\boldsymbol{a}$ and $\boldsymbol{x}$ is not fixed, as $\boldsymbol{a}$ can be a point representation and $\boldsymbol{x}$ can be line/plane coefficients. It means that those geometric primitives are *dual* [2].

The principle of duality in E^2 states that any theorem remains true when we interchange the words "point" and "line", "lie on" and "pass through", "join" and "intersection" and so on. Once the theorem has been established, the dual theorem is obtained as described above, see [3].

In other words, the principle of duality in E^2 says that in all theorems it is possible to substitute the term "point" by the term "line" and the term "line" by the term "point" and the given theorem stays valid. This helps a lot in solving some geometrical cases. In the case of E^3 point is a dual to a plane and vice versa etc..

Let us consider two simple problems in E^2:

- a line is given as a join of two points – this leads to a system of linear equations $\boldsymbol{Ax} = \boldsymbol{0}$
- a point given as the intersection of two lines – this leads to a system of linear equations $\boldsymbol{Ax} = \boldsymbol{b}$

If solved in the Euclidean space two significantly different problems are obtained. However the principle of duality leads to a natural question:

Why two different problems are obtained if the given problems are dual?

Similarly in E^3 a plane is given as a join of three points and a point is given as an intersection of three planes.

An elegant, robust and fast solution is presented in the chapter 4.

3 Linear and Spherical Interpolation in Projective Space

Interpolation is very often used in geometrical problems; mostly linear interpolation in the Euclidean space is used.

Linear Interpolation with Linear Parameterization

Linear interpolation in the Euclidean d-dimensional space is defined as:

$$\boldsymbol{X}(t) = \boldsymbol{X}_1 + (\boldsymbol{X}_2 - \boldsymbol{X}_1)\, t \qquad (8)$$

and has *linear parameterization*. If barycentric interpolation is used on a d-simplex:

$$\boldsymbol{X}(\boldsymbol{\lambda}) = \sum_{i=1}^{d} \lambda_i \boldsymbol{X}_i \qquad \sum_{i=1}^{d} \lambda_i = 1 \qquad (9)$$

where: $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_d)$, i.e. in E^2:

$$\boldsymbol{X}(\lambda_1, \lambda_2) = \lambda_1 \boldsymbol{X}_1 + \lambda_2 \boldsymbol{X}_2 \qquad \lambda_1 + \lambda_2 = 1 \qquad (10)$$

and in E^3:

$$\boldsymbol{X}(\lambda_1, \lambda_2, \lambda_3) = \lambda_1 \boldsymbol{X}_1 + \lambda_2 \boldsymbol{X}_2 + \lambda_3 \boldsymbol{X}_3 \qquad \lambda_1 + \lambda_2 + \lambda_3 = 1 \qquad (11)$$

The question is what happens, if points $\boldsymbol{X}_i$ are given in the homogeneous coordinates.

Linear Interpolation with Monotonically Parameterized

If points are given in homogeneous coordinates, linear interpolation can be made directly in the projective space, i.e. $\boldsymbol{x} = [x, y: w]^T$. In this case:

$$\boldsymbol{x}(t) = \boldsymbol{x}_1 + (\boldsymbol{x}_2 - \boldsymbol{x}_1)\, t \qquad (12)$$

i.e.

$$\begin{matrix} x(t) = x_1 + (x_2 - x_1)\, t & y(t) = y_1 + (y_2 - y_1)\, t \\ z(t) = z_1 + (z_2 - z_1)\, t & w(t) = w_1 + (w_2 - w_1)\, t \end{matrix} \qquad (13)$$

It is again a linear interpolation. However, the coordinates $\boldsymbol{X}$ changes with the value of t *non-linearly, but monotonically* [6], [7]. This property can be used for efficient algorithms deciding which point is closer to an observer etc. Computation of the barycentric coordinates in with homogeneous coordinates is similar [6].

4 Computation in Projective space

Let us consider following simple examples demonstrating the proposed approaches.

Intersection Operation

Let us consider a simple example when a point is given as an intersection of two lines $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$. This leads to a system of linear equations $\boldsymbol{Ax} = \boldsymbol{b}$ with a usual solution:

$$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} -d_1 \\ -d_2 \end{bmatrix} \qquad X = \frac{Det_x}{Det} \qquad Y = \frac{Det_y}{Det} \tag{14}$$

But there is question what happens, if the value of Det is small, i.e. when $|Det| < \varepsilon$. It does not generally mean that the given lines are parallel or close to parallel!

It can be proved that the intersection point $\boldsymbol{x}$ of two lines $\boldsymbol{p}_1 = [a_1, b_1 : d_1]^T$ and $\boldsymbol{p}_2 = [a_2, b_2 : d_2]^T$ can be easily computed using the cross product as:

$$\boldsymbol{x} = \boldsymbol{p}_1 \times \boldsymbol{p}_2 = det \begin{bmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \end{bmatrix} = [x, y : w]^T \tag{15}$$

In the case of E^3 the intersection point $\boldsymbol{x}$ of three planes $\boldsymbol{\rho}_1$, $\boldsymbol{\rho}_2$ and $\boldsymbol{\rho}_3$ is given as:

$$\boldsymbol{x} = \boldsymbol{\rho}_1 \times \boldsymbol{\rho}_2 \times \boldsymbol{\rho}_3 = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} & \boldsymbol{l} \\ a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{vmatrix} = [x, y, z : w]^T \tag{16}$$

where: $\boldsymbol{\rho}_1 = [a_1, b_1, c_1 : d_1]^T$, $\boldsymbol{\rho}_2 = [a_2, b_2, c_2 : d_2]^T$, $\boldsymbol{\rho}_3 = [a_3, b_3, c_3 : d_3]^T$.

Join Operation

Let us consider a simple example of a line $\boldsymbol{p}$ given by two points $\boldsymbol{X}_1$ and $\boldsymbol{X}_2$. This leads to a system of linear equations $\boldsymbol{Ax} = \boldsymbol{0}$, i.e.:

$$\begin{bmatrix} X_1 & Y_1 & 1 \\ X_2 & Y_2 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{17}$$

It means that one parametric set of solution is obtained. *How to solve it?* The answer is simple due to the principle of duality, as a point and a line are dual in E^2. Therefore the line $\boldsymbol{p}$ given by two points is determined as:

$$\boldsymbol{p} = \boldsymbol{x}_1 \times \boldsymbol{x}_2 = det \begin{bmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ x_1 & y_1 & w_1 \\ x_2 & y_2 & w_2 \end{bmatrix} = [a, b : d]^T \tag{18}$$

In the case of E^3 a plane given by three points is given as:

$$\boldsymbol{\rho} = \boldsymbol{x}_1 \times \boldsymbol{x}_2 \times \boldsymbol{x}_3 = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} & \boldsymbol{l} \\ x_1 & y_1 & z_1 & w_1 \\ x_2 & y_2 & z_2 & w_2 \\ x_3 & y_3 & z_3 & w_3 \end{vmatrix} = [a, b, c : d]^T \tag{19}$$

It can be seen that the principle of duality applied with projective representation can lead to new, more stable and robust formula especially convenient for vector-vector architectures [5].

It means that *no division operation is needed* in the both cases.

5 Geometric Transformations

Geometric transformations are used in virtual reality and haptic systems and their efficiency and robustness must be considered. However it is necessary to note that *geometric transformations of points in homogeneous coordinates differ from transformation of implicitly defined objects*, i.e. lines, planes, and normal vectors, etc.

Geometric Transformations of Points

Geometric transformations of points are based matrix-vector multiplication, i.e. $\boldsymbol{x}' = \boldsymbol{T}\boldsymbol{x}$, where: $\boldsymbol{x} = [x, y, z: w]^T$ and matrix $\boldsymbol{T}$ is (4×4) size [4].

Geometric Transformations of Implicitly Defined Objects

In graphical applications positions of points are changed by an interaction etc.:

$$\boldsymbol{x}' = \boldsymbol{T}\boldsymbol{x} \qquad (20)$$

The question is how coefficients of a line $\boldsymbol{p}$, resp. of a plane $\boldsymbol{\rho}$ are changed if points are transformed without a need of computation lines or planes from their definition.

It can be proved that:

$$\boldsymbol{p}' = (\boldsymbol{T}\boldsymbol{x}_1) \times (\boldsymbol{T}\boldsymbol{x}_2) = det(\boldsymbol{T})(\boldsymbol{T}^{-1})^T\boldsymbol{p} \triangleq (\boldsymbol{T}^{-1})^T\boldsymbol{p} = [a', b': d']^T \qquad (21)$$

or

$$\begin{aligned}\boldsymbol{\rho}' = (\boldsymbol{T}\boldsymbol{x}_1) \times (\boldsymbol{T}\boldsymbol{x}_2) \times (\boldsymbol{T}\boldsymbol{x}_3) = det(\boldsymbol{T})(\boldsymbol{T}^{-1})^T\boldsymbol{\rho} \triangleq (\boldsymbol{T}^{-1})^T\boldsymbol{\rho} \\ = [a', b', c': d']^T\end{aligned} \qquad (22)$$

where: $\triangleq$ means protectively equivalent.

It means that *transformation matrix of a normal vector is generally different from the matrix for transformation of points*.

6 Solution of Selected Problems

There are nice examples how the projective representation can simplify a solution of geometrical problems [8].

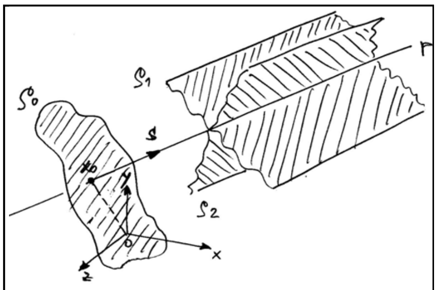

Fig.1: Intersection of two planes

Line in E^3 as Two Plane Intersection

Let us consider a "standard" formula first. Planes $\boldsymbol{\rho}_1$ and $\boldsymbol{\rho}_2$ are defined as:

$$\boldsymbol{\rho}_1 = [a_1, b_1, c_1: d_1]^T = [\boldsymbol{n}_1^T: d_1]^T \quad \boldsymbol{\rho}_2 = [a_2, b_2, c_2: d_2]^T = [\boldsymbol{n}_2^T: d_2]^T \qquad (23)$$

In the Euclidean space a line given as an intersection of two planes is given as:

$$\boldsymbol{s} = \boldsymbol{n}_1 \times \boldsymbol{n}_2 \equiv [a_3, b_3, c_3]^T \qquad \boldsymbol{x}(t) = \boldsymbol{x}_0 + \boldsymbol{s}t \qquad (24)$$

and

$$DET = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \qquad x_0 = \frac{d_2 \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} - d_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}}{DET}$$

$$y_0 = \frac{d_2 \begin{vmatrix} a_3 & c_3 \\ a_1 & c_1 \end{vmatrix} - d_1 \begin{vmatrix} a_3 & c_3 \\ a_2 & c_2 \end{vmatrix}}{DET} \qquad z_0 = \frac{d_2 \begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix} - d_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}}{DET} \qquad (25)$$

Line in E3 as Two Plane Intersection – Projective Solution

Directional $\boldsymbol{s}$ vector of a line given by two planes is $\boldsymbol{s} = \boldsymbol{n}_1 \times \boldsymbol{n}_2$. Let us consider a plane $\boldsymbol{\rho}_0$ passing the origin with the normal $\boldsymbol{s}$ vector, i.e. $\boldsymbol{\rho}_0 = [a_0, b_0, c_0 : 0]^T$. Then the "starting" point $\boldsymbol{x}_0$ can be determined as:

$$\boldsymbol{x}_0 = \boldsymbol{\rho}_1 \times \boldsymbol{\rho}_2 \times \boldsymbol{\rho}_0 \qquad (26)$$

How simple the formula is!

7 Conclusion

This paper describes new approaches to solutions of geometrical problems related to applications, especially applicable in virtual reality and haptic systems based on projective space representation. Shortly introduced principle of duality enables solving dual problems by one sequence resulting to more robust code. As the projective representation supports vector-vector operations naturally, application of GPU architecture or usage of SSE instructions is natural and significant speed up can be expected as well. If homogeneous representation is used, division operation is postponed in principle and a higher precision of computation can be expected as well.

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