











$$\begin{aligned}
 & -[s_x, s_y, 0]^T \begin{pmatrix} x_A \\ y_A \\ z_C \end{pmatrix} - R \begin{pmatrix} \cos\varphi \\ 0 \\ \sin\varphi \end{pmatrix} \\
 & = s_x(R\cos\varphi - x_A)
 \end{aligned}$$

The point  $\mathbf{x}(t_{extrem}(\varphi))$  is inside of the envelope; see Fig.4, if and only if  $F(t_{extrem}(\varphi)) < 0$ . Substituting  $t_{extrem}$  to the function  $F(t)$

$$t^2 + 2\mathbf{s}^T \boldsymbol{\xi}(\varphi)t + \boldsymbol{\xi}^T(\varphi)\boldsymbol{\xi}(\varphi) - r^2 = 0 \tag{61}$$

we get

$$\begin{aligned}
 & (-\mathbf{s}^T \boldsymbol{\xi}(\varphi))^2 \\
 & + (-\mathbf{s}^T \boldsymbol{\xi}(\varphi) \cdot 2\mathbf{s}^T \boldsymbol{\xi}(\varphi)) \\
 & + \boldsymbol{\xi}^T(\varphi)\boldsymbol{\xi}(\varphi) - r^2 = 0
 \end{aligned} \tag{62}$$

i.e.

$$-\mathbf{s}^T \boldsymbol{\xi}(\varphi) + \boldsymbol{\xi}^T(\varphi)\boldsymbol{\xi}(\varphi) - r^2 = 0 \tag{63}$$

Substituting

$$\begin{aligned}
 \mathbf{s} &= [s_x, s_y, 0]^T \\
 \boldsymbol{\xi}(\varphi) &= \mathbf{x}_A - \mathbf{x}_s(\varphi)
 \end{aligned} \tag{64}$$

This leads to:

$$\boldsymbol{\xi}^T(\varphi)\boldsymbol{\xi}(\varphi) - (\mathbf{s}^T \boldsymbol{\xi}(\varphi))^2 = r^2 \tag{65}$$

Therefore

$$\boldsymbol{\xi}^T(\varphi)[\mathbf{I} - \mathbf{s}^T \otimes \mathbf{s}]\boldsymbol{\xi}(\varphi) = 0 \tag{66}$$

where  $\mathbf{I}$  is an identity matrix and  $\otimes$  is a tensor product producing a matrix.

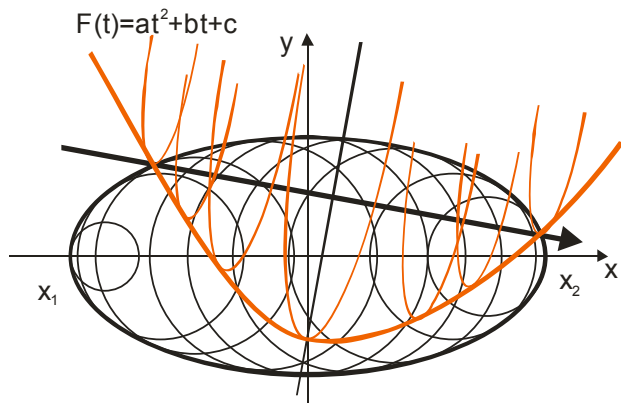


Figure 4: Rotating sphere plane intersection and envelope

As we recently set  $a = 1$  in the quadratic equation, we can write

$$t_1(\varphi)t_2(\varphi) = c(\varphi) \tag{67}$$

where  $t_1(\varphi)$  and  $t_2(\varphi)$  are the line parameter values for line sphere intersection.

The second Vieta's [2] equation can be used to determine intervals for  $\varphi$  with one root only for iterative solvers.

In the classified case:

- ad a) we can use mirroring operations and solve the intersection in one quadrant only twice for non-mirrored and for mirrored cases as there might be two tuples of intersections,

- ad b) situation is complex as the envelope has an inflection point so there might be three intersections in one quadrant
- ad c) this case is similar to the previous but only two intersection points might occur

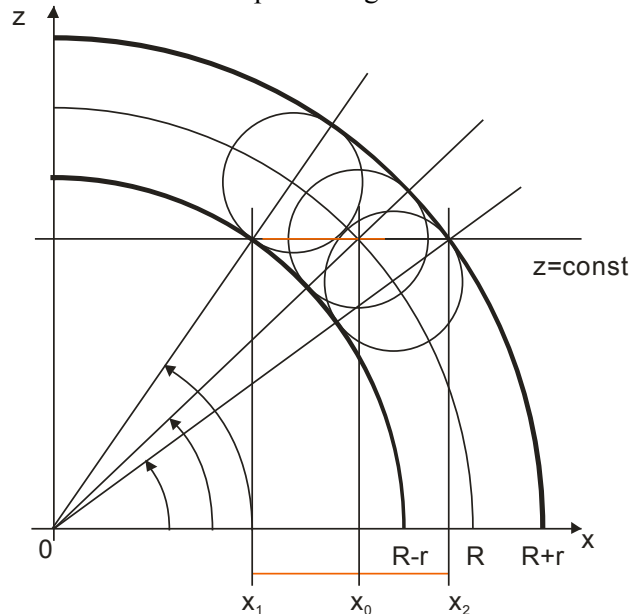


Figure 5: Rotating spheres

However the intersection computation is still too complex.

### 3.2 Line Rotation – Intersection in $E^3$

Another alternative approach is based on a fixed sphere position on the  $x$  axis and the given line rotates about  $y$  axis generally in  $E^3$ . This approach is actually “dual” in some sense to the previous one and leads to an envelope given as an intersection of a sphere and double cone.

There are two possible equivalent formulations: the center of a sphere is on the  $x$  axis and the rotating line is in a general position in  $E^3$  or geometric transformation is made so that the rotating line rotates about  $y$  axis and the vertex of a double cone is in the origin of the coordinate system; the center of a sphere is in the  $x - y$  plane, i.e. was moved up.

A line in  $E^3$  is defined as

$$\begin{aligned}
 \mathbf{x}(t) &= \mathbf{x}_A + \mathbf{s}t \\
 \mathbf{s} &= [s_x, s_y, 0]^T
 \end{aligned} \tag{68}$$

and a sphere on the  $x$  axis is defined as

$$\begin{aligned}
 (\mathbf{x} - \mathbf{x}_s)^T(\mathbf{x} - \mathbf{x}_s) - r^2 &= 0 \\
 \mathbf{x}_s &= [R, 0, 0]^T
 \end{aligned} \tag{69}$$

As the line is rotated about  $y$  axis the rotation matrix  $\mathbf{R}$  is expressed as

$$\mathbf{R}(\varphi) = \begin{bmatrix} \cos\varphi & 0 & -\sin\varphi \\ 0 & 1 & 0 \\ \sin\varphi & 0 & \cos\varphi \end{bmatrix} \quad (70)$$

Then the rotating line forming a double cone in  $E^3$  can be expressed as

$$\mathbf{x}(t, \varphi) = \mathbf{R}(\varphi)(\mathbf{x}_A + \mathbf{s} t) \quad (71)$$

Substituting we get

$$\begin{aligned} &(\mathbf{R}(\varphi) \mathbf{x}_A + \mathbf{R}(\varphi) \mathbf{s} t - \mathbf{x}_s)^T \\ &(\mathbf{R}(\varphi) \mathbf{x}_A + \mathbf{R}(\varphi) \mathbf{s} t - \mathbf{x}_s) - r^2 = 0 \end{aligned} \quad (72)$$

or

$$\begin{aligned} &(\mathbf{R}(\varphi) \mathbf{s} t + \boldsymbol{\xi})^T (\mathbf{R}(\varphi) \mathbf{s} t + \boldsymbol{\xi}) - r^2 \\ &= 0 \\ &\boldsymbol{\xi}(\varphi) = \mathbf{R}(\varphi) \mathbf{x}_A - \mathbf{x}_s \end{aligned} \quad (73)$$

It means that a quadratic equation is obtained again, i.e.

$$\begin{aligned} &\mathbf{s}^T \mathbf{R}^T(\varphi) \mathbf{R}(\varphi) \mathbf{s} t^2 \\ &+ 2\mathbf{s}^T \mathbf{R}^T(\varphi) \boldsymbol{\xi}(\varphi) + \boldsymbol{\xi}^T(\varphi) \boldsymbol{\xi}(\varphi) \\ &= 0 \end{aligned} \quad (74)$$

As the matrix  $\mathbf{R}(\varphi)$  is orthonormal, i.e.  $\mathbf{R}^T(\varphi) \mathbf{R}(\varphi) = \mathbf{I}$  and directional vector can be normalized, i.e.  $\mathbf{s}^T \mathbf{s} = 1$  then we get a significant simplification

$$\begin{aligned} &t^2 + 2\mathbf{s}^T \mathbf{R}^T(\varphi) \boldsymbol{\xi}(\varphi) t \\ &+ \boldsymbol{\xi}^T(\varphi) \boldsymbol{\xi}(\varphi) = 0 \end{aligned} \quad (75)$$

Let us explore coefficients of this quadratic equation more in a detail.

$$\begin{aligned} &\mathbf{s}^T \mathbf{R}^T(\varphi) \boldsymbol{\xi}(\varphi) \\ &= \mathbf{s}^T \mathbf{R}^T(\varphi) (\mathbf{R}(\varphi) \mathbf{x}_A - \mathbf{x}_s) \\ &= \mathbf{s}^T \mathbf{R}^T(\varphi) \mathbf{R}(\varphi) \mathbf{x}_A - \mathbf{s}^T \mathbf{R}^T(\varphi) \mathbf{x}_s \end{aligned} \quad (76)$$

As  $\mathbf{R}^T(\varphi) \mathbf{R}(\varphi) = \mathbf{I}$  we get

$$\begin{aligned} &\mathbf{s}^T \mathbf{R}^T(\varphi) \boldsymbol{\xi}(\varphi) = \\ &\mathbf{s}^T \mathbf{x}_A - \mathbf{s}^T \mathbf{R}^T(\varphi) \mathbf{x}_s \end{aligned} \quad (77)$$

Using cross product symmetry we get

$$\begin{aligned} &\mathbf{s}^T \mathbf{R}^T(\varphi) \boldsymbol{\xi}(\varphi) = \\ &\mathbf{s}^T \mathbf{x}_A - \mathbf{x}_s^T \mathbf{R}(\varphi) \mathbf{x}_s \end{aligned} \quad (78)$$

Now there is another simplification possible as  $\mathbf{x}_s = [R, 0, 0]^T$  and  $\mathbf{s} = [s_x, s_y, 0]^T$

$$\begin{aligned} &\mathbf{s}^T \mathbf{x}_A - \mathbf{x}_s^T \mathbf{R}(\varphi) \mathbf{x}_s \\ &= [s_x, s_y, 0] [x_A, y_A, z_A]^T \\ &- [R, 0, 0] \begin{bmatrix} \cos\varphi & 0 & -\sin\varphi \\ 0 & 1 & 0 \\ \sin\varphi & 0 & \cos\varphi \end{bmatrix} \begin{bmatrix} R \\ 0 \\ 0 \end{bmatrix} \\ &= s_x x_A + s_y y_A - R^2 \cos\varphi \end{aligned} \quad (79)$$

Now the last term of the equation

$$\begin{aligned} &\boldsymbol{\xi}^T(\varphi) \boldsymbol{\xi}(\varphi) = \\ &[\mathbf{R}(\varphi) \mathbf{x}_A - \mathbf{x}_s]^T [\mathbf{R}(\varphi) \mathbf{x}_A - \mathbf{x}_s] \\ &= \mathbf{x}_A^T \mathbf{R}^T(\varphi) \mathbf{R}(\varphi) \mathbf{x}_A \\ &- 2\mathbf{x}_A^T \mathbf{R}^T(\varphi) \mathbf{x}_s + \mathbf{x}_s^T \mathbf{x}_s \end{aligned} \quad (80)$$

As  $\mathbf{R}^T(\varphi) \mathbf{R}(\varphi) = \mathbf{I}$

$$\begin{aligned} &\boldsymbol{\xi}^T(\varphi) \boldsymbol{\xi}(\varphi) = \\ &= \mathbf{x}_A^T \mathbf{x}_A - \mathbf{x}_A^T \mathbf{R}^T(\varphi) \mathbf{x}_s + \mathbf{x}_s^T \mathbf{x}_s \\ &= \mathbf{x}_A^T \mathbf{x}_A - \mathbf{x}_s^T \mathbf{R}(\varphi) \mathbf{x}_A + \mathbf{x}_s^T \mathbf{x}_s \end{aligned} \quad (81)$$

Using cross product symmetry we get

$$\begin{aligned} &\boldsymbol{\xi}^T(\varphi) \boldsymbol{\xi}(\varphi) = \\ &\mathbf{x}_A^T \mathbf{x}_A - 2\mathbf{x}_s^T \mathbf{R}(\varphi) \mathbf{x}_A + \mathbf{x}_s^T \mathbf{x}_s \end{aligned} \quad (82)$$

Again, there is another simplification possible as  $\mathbf{x}_s = [R, 0, 0]^T$  and  $\mathbf{s} = [s_x, s_y, 0]^T$

$$\begin{aligned} &\boldsymbol{\xi}^T(\varphi) \boldsymbol{\xi}(\varphi) = \\ &\mathbf{x}_A^T \mathbf{x}_A - 2\mathbf{x}_s^T \mathbf{R}(\varphi) \mathbf{x}_A + \mathbf{x}_s^T \mathbf{x}_s \\ &= \mathbf{x}_A^T \mathbf{x}_A \\ &- 2[R, 0, 0] \begin{bmatrix} \cos\varphi & 0 & -\sin\varphi \\ 0 & 1 & 0 \\ \sin\varphi & 0 & \cos\varphi \end{bmatrix} \begin{bmatrix} x_A \\ y_A \\ z_A \end{bmatrix} \\ &+ R^2 \\ &= \mathbf{x}_A^T \mathbf{x}_A - 2R(x_A \cos\varphi - z_A \sin\varphi) \\ &+ R^2 \end{aligned} \quad (83)$$

Putting all together we get

$$\begin{aligned} &t^2 + 2(\mathbf{s}^T \mathbf{x}_A - \mathbf{x}_s^T \mathbf{R}(\varphi) \mathbf{x}_s) t \\ &- 2\mathbf{x}_s^T \mathbf{R}(\varphi) \mathbf{x}_A + \mathbf{x}_A^T \mathbf{x}_A + \mathbf{x}_s^T \mathbf{x}_s = 0 \end{aligned} \quad (84)$$

i.e.

$$\begin{aligned} &t^2 + (s_x x_A + s_y y_A - R^2 \cos\varphi) t + \mathbf{x}_A^T \mathbf{x}_A \\ &- 2R(x_A \cos\varphi \\ &- z_A \sin\varphi) + R^2 = 0 \end{aligned} \quad (85)$$

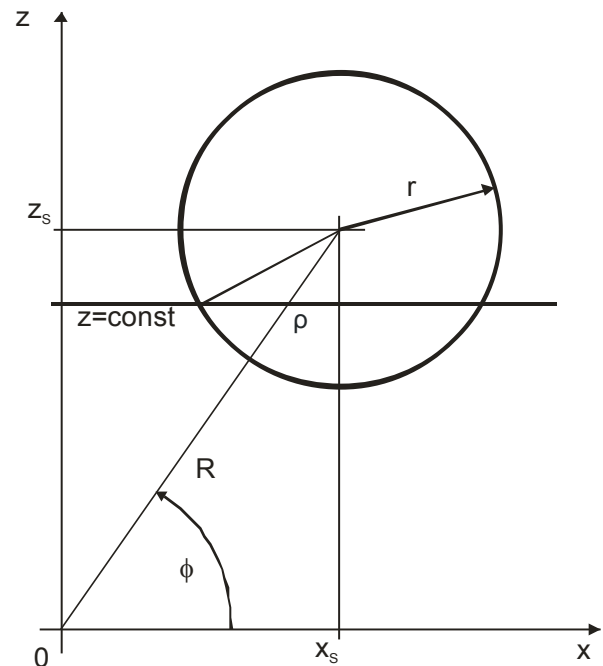


Figure 6: Intersection plane-rotating sphere

### 3.3 Intersection with a Plane - Solution in $E^2$

In this part we will concentrate on the case, when sphere rotates about  $y$  axis and intersect a plane on which the given line lies and is parallel to the  $x - z$  plane

As the given line lies in a plane parallel to the  $x - z$  plane the rotating sphere intersect the plane, Fig.5, which results into circles in the  $x - z$  plane, Fig.6.

$$(\mathbf{x} - \mathbf{x}_s)^T(\mathbf{x} - \mathbf{x}_s) - \rho^2 = 0 \quad (86)$$

Let us consider the line formulation.

$$\begin{aligned} (\mathbf{x} - \mathbf{x}_s(\varphi))^T(\mathbf{x} - \mathbf{x}_s(\varphi)) - r^2 &= 0 \\ \mathbf{x}_s(\varphi) &= R[\cos\varphi, 0, \sin\varphi]^T \end{aligned} \quad (87)$$

A sphere is rotating about  $y$  axis is described by i.e.

$$\begin{aligned} [x - x_s(\varphi), y, z - z_s(\varphi)]^T \\ [x - x_s(\varphi), y, z - z_s(\varphi)] - r^2 &= 0 \end{aligned} \quad (88)$$

A plane on which the given line lies is defined as  $z = z_c$ . Then

$$\begin{aligned} (x - x_s)^2 + y^2 + (z - z_s)^2 - r^2 \\ = x^2 - 2xR\cos(\varphi) + R^2\cos^2(\varphi) \\ + y^2 \\ + z_c^2 - 2z_cR\sin(\varphi) + R^2\sin^2(\varphi) \\ - r^2 &= 0 \end{aligned} \quad (89)$$

As  $\cos^2(\varphi) + \sin^2(\varphi) = 1$  we get

$$\begin{aligned} (x - x_s)^2 + y^2 + (z - z_s)^2 - r^2 \\ = x^2 + y^2 - 2R(x\cos(\varphi) \\ + z_c\sin(\varphi)) \\ + z_c^2 + R^2 - r^2 &= 0 \end{aligned} \quad (90)$$

As the given line is defined as

$$\begin{aligned} x(t) &= x_A + s_x t \\ y(t) &= y_A + s_y t \end{aligned} \quad (91)$$

we get

$$\begin{aligned} x_A^2 + 2x_A s_x t + s_x^2 t^2 + \\ y_A^2 + 2y_A s_y t + s_y^2 t^2 \\ - 2R((x_A + s_x t)\cos(\varphi) + z_c\sin(\varphi)) \\ + z_c^2 + R^2 - r^2 &= 0 \end{aligned} \quad (92)$$

i.e. a quadratic equation has a form

$$\begin{aligned} (s_x^2 + s_y^2)t^2 \\ + 2[(x_A - R\cos(\varphi))s_x + y_A s_y]t \\ - 2R(x_A\cos(\varphi) + z_c\sin(\varphi)) \\ + x_A^2 + y_A^2 + z_c^2 + R^2 - r^2 &= 0 \end{aligned} \quad (93)$$

In the case of the normalized directional vector  $\mathbf{s}$ , i.e.  $\|\mathbf{s}\| = 1$ , resp.  $\mathbf{s}^T \mathbf{s} = 1$ , we get a quadratic equation parameterized by  $\varphi$  as follows

$$\begin{aligned} t^2 + bt + c &= 0 \\ b &= 2[(x_A - R\cos(\varphi))s_x + y_A s_y] \\ c &= -2R(x_A\cos(\varphi) + z_c\sin(\varphi)) \\ + x_A^2 + y_A^2 + z_c^2 + R^2 - r^2 \end{aligned} \quad (94)$$

### 3.4 Hybrid method

Let torus is represented as an envelope of rotating spheres about  $y$  axis again. Spheres intersect the plane  $z = z_c$  on which the given line lies and form circles in the plane  $z = z_c$ , on the plane parallel to  $z - y$  plane. Those circles on the plane are described by an equation

$$(x - x_s)^2 + y^2 - \rho^2 = 0 \quad (95)$$

where

$$x_s = R\cos\varphi \quad y_s = 0 \quad z_s = R\sin\varphi \quad (96)$$

Note that  $z_s$  represents rotation of the sphere about  $y$  axis, resulting circle is on the  $z = z_c$  plane. The  $\rho$  radius of a circle is given

$$\begin{aligned} \rho^2 &= r^2 - (z_s - z_c)^2 = \\ r^2 - (R\sin\varphi - z_c)^2 \end{aligned} \quad (97)$$

The envelope of a plane-torus intersection is given as

$$\bigcup_{\varphi \in \langle \varphi_1, \varphi_2 \rangle} \{(x - R\cos\varphi)^2 + y^2 - r^2 + (R\sin\varphi - z_c)^2 = 0\} \quad (98)$$

Let us consider the case, when  $z_c < R - r$ , Fig.7.

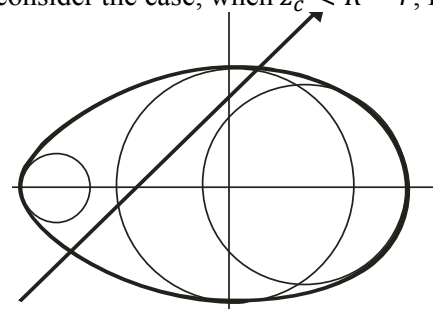


Figure 7: An envelope given as union of plane-rotating sphere intersections

Angles are determined as follows

$$\begin{aligned} \cos(\varphi_1) &= (R - r)^2 - z_c^2 \\ \cos(\varphi_2) &= (R + r)^2 - z_c^2 \\ \cos(\varphi_0) &= R^2 - z_c^2 \end{aligned} \quad (99)$$

The angle  $\varphi_1$  is an angle when the first circle that contributes to an envelope; the angle  $\varphi_2$  is for the last circle that contributes to the envelope and the angle  $\varphi_0$  is for the largest circle inside the envelope.

The given line lies in the  $z = z_c$  plane and is defined as

$$\begin{aligned} x(t) &= x_A + s_x t \\ y(t) &= y_A + s_y t \end{aligned} \quad (100)$$

The line can be re-parameterized so that  $y_A = 0$  then circles are defined as:

$$\begin{aligned} x(x_A + s_x t - R\cos\varphi)^2 + s_y^2 t^2 \\ - r^2 + (R\sin\varphi - z_c)^2 &= 0 \end{aligned} \quad (101)$$

Now the problem is effectively transferred to  $E^2$ .



### 3.5 New Bounding Volume

The “standard” bounding volume [1] is based on a sphere in  $E^3$  and an intersection of two half spaces, Fig.2. As the line lies in the  $x - y$  plane for  $z = z_c$  we can distinguish following fundamental cases:

- ad a) we can use mirroring operations and solve the intersection in one quadrant only twice for non-mirrored and for mirrored cases as there might be two tuples of intersections,
- ad b) situation is complex as the envelope has an inflection point so there might be three intersections in one quadrant,
- ad c) this case is similar to the previous but only two intersection points might occur.

However if many lines-torus intersections computation are needed, like in the ray tracing rendering technique, the more precise bounding volume is needed to increase the efficiency of computation. The “standard” bounding volume works fine for the case ad b). On the other hand it can be seen that

- in the case ad a), i.e. when a line passes through the torus, i.e. through a “hole” and does not intersect the torus, detailed computation has to be made, that is computationally expensive.
- in the case ad c), i.e. when a line intersects the torus in its “outer part”, i.e.  $R \leq z_c < R + r$  the distance between two planes can be smaller than  $2r$ .

Let us explore the first case as it leads to higher efficiency.

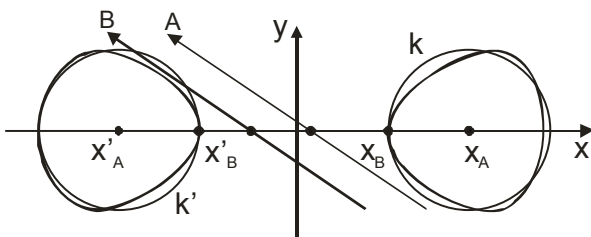


Figure 8: Torus-plane intersection and a ray

Fig.8 presents an intersection plane-torus for  $0 < z_c < R - r$ . It can be seen that a  $k$  circle (as we are in  $E^2$ ), with the center at  $x_A$  with the radius  $r$  forms bounding surfaces together with the mirrored  $k'$  circle by  $y$  axis. The  $x_A$  center of the circle is defined as follows:

$$\begin{aligned} x_B &= (R - r)\cos\varphi \\ x_A &= x_B + r = (R - r)\cos\varphi + r \end{aligned} \quad (102)$$

where

$$\sin\varphi = \frac{z_c}{R} \quad (103)$$

or

$$x_B = \sqrt{(R - r)^2 - z_c^2} \quad (104)$$

$$x_A = x_B + r = r + \sqrt{(R - r)^2 - z_c^2}$$

It can be seen that in the case of  $0 < z_c = R - r$  a special case is obtained as there is no “hole” at all, Fig.9

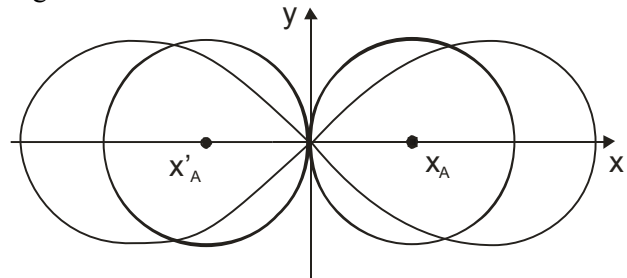


Figure 9: A boundary situation

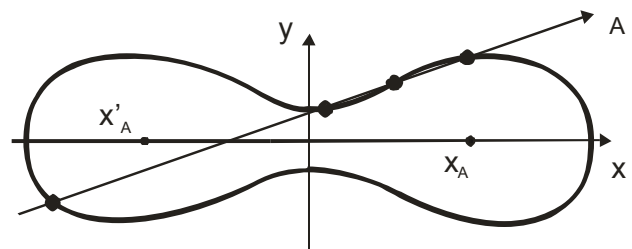


Figure 10: Line-torus intersection for  $R - r \leq z_c < R$ , i.e. the case ad b)

The test for the ad a) case can be formulated as: if the line intersects the  $x$  axis in the interval  $(x'_B, x_B)$  and does not intersect the circle  $k$  nor the circle  $k'$ , then the line does not intersect the given torus. Fig.6 presents two lines, in the case A, the line is not considered for intersection computation with torus, while in the cases B, the detailed intersection test/computation has to be made.

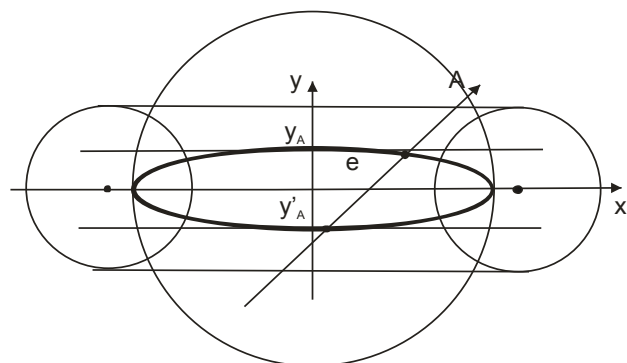


Figure 11: Line-torus intersection and bounding for  $R \leq z_c < R + r$ , i.e. the case ad c)

The test for the ad b) test remains as the original, Fig.10, as up to 3 intersections can occur in one quadrant as there is a point of inflexion.

In the case ad c), i.e.  $R \leq z_c < R + r$ , there are only 2 intersection points possible, Fig.11. It can be seen that the distance between two planes, given by  $y_A$  and  $y'_A$  values is now smaller than the original distance  $2r$ . It can be seen that the new distance is given as

$$d^2 = r^2 - (z_c - r)^2 \quad (105)$$

#### 4 Conclusion

New alternative formulations for line-torus intersection problem have been presented. Unfortunately all the presented alternative formulations do not lead to simpler computational formulas. It seems to that an implicit form for the line-torus intersection is the most efficient one. There is still one possibility to use toroidal coordinate system; however the computational expense is too high.

As a result of new geometrically equivalent formulations a new bounding object, actually circles in  $E^2$ , for the line-torus intersection has been developed and described.

The new bounding object increases line-torus intersection computation efficiency significantly as it also detects the cases when a line or ray is passing a "hole" of the torus. The efficiency of the new torus bounding test grows with the ratio  $\nu = R/r$ .

#### Acknowledgment

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