A New Torus Bounding for Line-Torus Intersection

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***Abstract*—Intersection algorithms are very important in computation of geometrical problems. An intersection of a line with linear or quadratic surfaces is well done, however a line intersection with other surfaces is more complex and time consuming. In this case the object is usually closed into a simple bounding volume to speed up the cases when the given line cannot intersect the given object. In this paper a new formulation of the line-torus intersection problem is given and new specification of the bounding volume for a torus is given. The presented approach is based on an idea of a line intersection with an envelope of rotating sphere that forms a torus. Due to this approach new bounding volume can be formulated which is more effective as it enables to detect cases when the line passes the "hole" of a torus.**

***Keywords*—torus intersection; bounding volume; intersection; line clipping; CAD systems**

I. INTRODUCTION

Intersection algorithms play a significant role in all geometric problems and CAD/CAM systems. Intersection algorithms are well documented for linear cases, e.g. line-plane or line-triangle etc., and also for some specific non-linear surfaces like line-sphere intersection etc. However, there are other objects like bicubic parametric patches, torus etc. In this case computation of intersection points is more complex and usually complex formula or iterative formula are to be used.

Intersection of a line and closed surface can be considered as generalized well known clipping problem. Intersection of a line or ray with a surface is the key problem solved in all ray-tracing techniques. Due to the computational complexity a bounding volumes are used to detect cases when a line cannot intersect the given object.

In this paper we present torus-line intersection problem, which leads to a quartic equation in principle, and show other possible formulations of line-torus intersection problem. These reformulations lead to a formulation of a new problem – generalized line clipping by an envelope (convex or non-convex) of parametric closed surfaces.

II. TORUS LINE INTERSECTION

A. Traditional approach

Torus-line intersection is actually a solution of a line in E^3 usually given in a parametric form as

$$\boldsymbol{x}(t) = \boldsymbol{x}_A + \boldsymbol{s}\,t \quad \boldsymbol{s} = [s_x, s_y, s_z]^T \qquad (1)$$

and a torus, which is generally a surface of the 4th order and can be given as :

$$(x^2 + y^2 + z^2 + R^2 - r^2)^2 = 4R^2(x^2 + y^2) \qquad (2)$$

Note that the z axis is rotational. This torus equation can be reformulated as

$$(\boldsymbol{x}^T\boldsymbol{x} + \xi)^2 = 4R\boldsymbol{x}^T\boldsymbol{M}\boldsymbol{x} \quad \boldsymbol{x} = [x, y, z]^T \qquad (3)$$

where

$$\xi = R^2 - r^2 \qquad \boldsymbol{M} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad (4)$$

Fig. 1. Torus

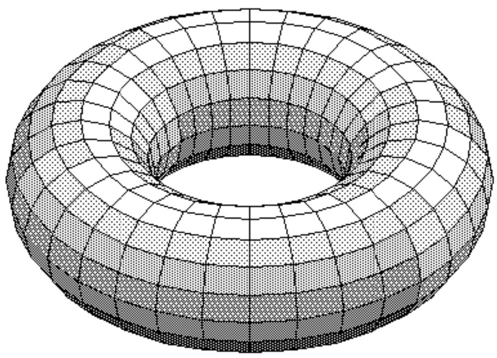
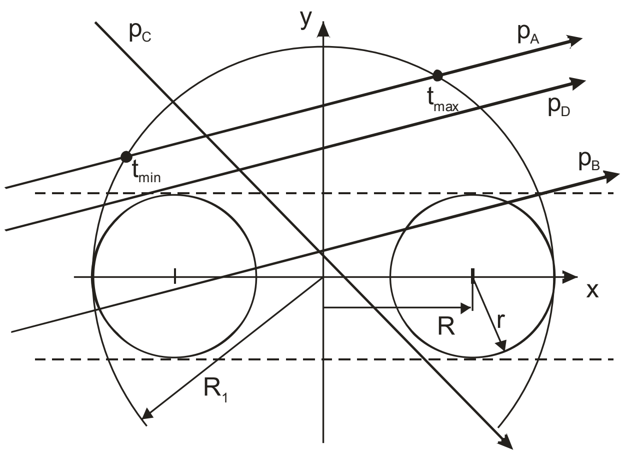


Fig. 2. Bounding Volume



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As there will be some geometric transformation used latter on we can also scale the given torus and a line so that $R = 1$, i.e. the torus is "normalized".

Now the intersection of a line and the torus is given as a solution of equations:

$$\boldsymbol{x}(t) = \boldsymbol{x}_A + \boldsymbol{s}\,t \tag{5}$$

and

$$(\boldsymbol{x}^T\boldsymbol{x} + \xi)^2 = 4\boldsymbol{x}^T\boldsymbol{M}\boldsymbol{x} \tag{6}$$

Substituting Eq.5 to Eq.6 we get

$$[(\boldsymbol{s}t + \boldsymbol{x}_A)^T(\boldsymbol{s}t + \boldsymbol{x}_A) + \xi]^2 = 4(\boldsymbol{s}t + \boldsymbol{x}_A)^T\boldsymbol{M}(\boldsymbol{s}t + \boldsymbol{x}_A) \tag{7}$$

and finally we get

$$[\boldsymbol{s}^T\boldsymbol{s}\,t^2 + 2\boldsymbol{s}^T\boldsymbol{x}_A\,t + \boldsymbol{x}_A^T\boldsymbol{x}_A + \xi]^2 - 4[\boldsymbol{s}^T\boldsymbol{M}\boldsymbol{s}\,t^2 + 2\boldsymbol{s}^T\boldsymbol{M}\boldsymbol{x}_A\,t + \boldsymbol{x}_A^T\boldsymbol{M}\boldsymbol{x}_A] = 0 \tag{8}$$

This equations is quite complex, but by detailed evaluation we get a quartic equation

$$\begin{gathered} at^4 + bt^3 + ct^2 + dt + e = 0 \\ \alpha = \boldsymbol{s}^T\boldsymbol{s} \qquad \beta = \boldsymbol{s}^T\boldsymbol{x}_A \qquad \gamma = \boldsymbol{x}_A^T\boldsymbol{x}_A \\ \delta = (\gamma + R^2 - r^2) \\ \sigma = (\gamma - R^2 - r^2) \qquad a = \alpha^2 \qquad b = 4\alpha\beta \\ c = 2\alpha(\gamma + R^2 - r^2) - 4R^2(s_x^2 + s_y^2) + 4\beta^2 \\ d = 8R^2 z_A s_z + 4\beta\sigma \\ e = \alpha^2 + (R^2 - r^2)^2 + 2\begin{bmatrix} s_x^2 s_y^2 + s_z^2(R^2 - r^2) \\ +(s_x^2 + s_y^2)(s_z^2 - R^2 - r^2) \end{bmatrix} \end{gathered} \tag{10}$$

It can be seen that the computation can be simplified for the case, when $\boldsymbol{s}^T\boldsymbol{s} = 1$, i.e. the directional vector of the line is normalized or the equation is divided by a . It means that we are getting a quartic equation on the form [3], [4].

$$t^4 + bt^3 + ct^2 + dt + e = 0 \tag{11}$$

which can be simplified by substitution

$$t = x - \frac{b}{4} \tag{12}$$

to

$$x^3 + px^2 + qx + r = 0 \tag{13}$$

where

$$\begin{gathered} p = \tfrac{3}{8}b^2 + c \qquad q = \tfrac{1}{8}b^3 - \tfrac{1}{2}bc + d \\ r = -\frac{3}{256}b^4 + \frac{1}{16}b^2c - \frac{1}{14}bd + e \end{gathered} \tag{14}$$

If solution for x is found then the solution of the original equation is given by Eq.12. To get a solution for x the following qubic equation has to be solved

$$\xi^3 - \frac{p}{2}\xi^2 - r\xi + \frac{4rp - q^2}{8} = 0 \tag{15}$$

Then the x values can be computed from real solution of the equation above as two quadratic equations as follows:

If $q \geq 0$ then

$$\begin{gathered} x^2 + x\sqrt{2\xi - p} + \xi - \sqrt{\xi^2 - r} = 0 \\ x^2 - x\sqrt{2\xi - p} + \xi + \sqrt{\xi^2 - r} = 0 \end{gathered} \tag{16}$$

If $q < 0$ then

$$\begin{gathered} x^2 + x\sqrt{2\xi - p} + \xi + \sqrt{\xi^2 - r} = 0 \\ x^2 - x\sqrt{2\xi - p} + \xi - \sqrt{\xi^2 - r} = 0 \end{gathered} \tag{17}$$

It can be seen that the solution itself is not simple, but the formula is closed. On the opposite, an iterative method like Bisection or Newton method can be used. However there are up to 4 intersections of the line and the torus, so it is necessary to find intervals for t , with one intersection only.

B. *Alternative Torus Representation*

There are other formulations of a torus as follows, but they are not convenient for our purposes.

$$\left(R - \sqrt{x^2 + y^2}\right)^2 + z^2 = r^2 \tag{18}$$

or a parametric form as

$$\begin{gathered} x(\varphi, \vartheta) = (R + r\cos\vartheta)\cos\varphi \\ y(\varphi, \vartheta) = (R + r\cos\vartheta)\sin\varphi \\ z(\varphi, \vartheta) = r\,\sin\vartheta \end{gathered} \tag{19}$$

It can be seen that a solution of a line and torus intersection is not a simple task and leads to a non-trivial computational problem.

However, there are some other geometrically equivalent formulations that could be used for finding a faster solution. In the following we will consider only a circular torus with a hope that the given approach can be extended also to more general cases as well, e.g. to an elliptical torus etc.

C. *Geometric Transformations*

There are other

Geometric transformations with points are defined in the projective space using homogeneous coordinates, i.e. in the projective extension of the Euclidean space.

A point $\boldsymbol{X} = (X, Y)$ in the Euclidean coordinates has homogeneous coordinates $\boldsymbol{x} = [x, y: w]^T$; w is the homogeneous coordinate. The conversion between the projective space and the Euclidean space is defined as

$$X = \frac{x}{w} \qquad y = \frac{x}{w} \qquad w \neq 0 \tag{20}$$

It means that the projective representation is actually a one parametric set. A point in the Euclidean space E^2 is represented as a line with the origin of the coordinate system excluded in the projective space. The origin represents a point in infinity. Geometric transformations with points like rotation, translation, mirroring etc. can be then described by the $\boldsymbol{Q}$ matrix [6], [10] as

$$\boldsymbol{x}' = \boldsymbol{Q}\boldsymbol{x} \tag{21}$$

Note that x, y coordinates might have some physical meaning and units, e.g. [m], while w has no unit, it is just a "scaling factor". That's why we used ":" to separate the values in the vector notation.

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A line in E^2 determined by two points given in the homogeneous coordinates can be computed using the cross product as [7], [8].

$$\boldsymbol{p} = \boldsymbol{x}_1 \times \boldsymbol{x}_2 = \begin{bmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ x_1 & y_1 & w_1 \\ x_2 & y_2 & w_2 \end{bmatrix} \quad (22)$$

$$\boldsymbol{p} = [a, b: c]^T \qquad p:\ ax + by + c = 0$$

Intersection of two lines p_1 and p_2 in E^2 can be computes as

$$\boldsymbol{x} = \boldsymbol{p}_1 \times \boldsymbol{p}_2 = \begin{bmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{bmatrix} \quad (23)$$

$$\boldsymbol{x} = [x, y: w]^T$$

We can see that both computations are in the case E^2 "dual", i.e. line and pints are dual [9]. In the case E^3 point is dual to a plane and vice versa. It can be shown that a plane given by three points can be determined by the extended cross product as

$$\boldsymbol{\rho} = \boldsymbol{x}_1 \times \boldsymbol{x}_2 \times \boldsymbol{x}_3 = \begin{bmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} & \boldsymbol{l} \\ x_1 & y_1 & z_1 & w_1 \\ x_2 & y_2 & z_2 & w_2 \\ x_3 & y_3 & z_3 & w_3 \end{bmatrix} \quad (24)$$

$$\boldsymbol{\rho} = [a, b, c: d]^T \quad \rho:\ ax + by + cz + d = 0$$

Again, an intersection of three planes can be computed as, see [7], [10] for details

$$\boldsymbol{x} = \boldsymbol{\rho}_1 \times \boldsymbol{\rho}_2 \times \boldsymbol{\rho}_3 = \begin{bmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} & \boldsymbol{l} \\ a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{bmatrix} \quad (25)$$

$$\boldsymbol{x} = [x, y, z: w]^T$$

This approach is simple, easy to implement and convenient for GPU implementation as well.

However, matrix transformations for points cannot be used for geometric transformations with lines in E^2 nor with planes in E^3 [10]. It can be shown that if a line p given by two points $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ and those points are geometrically transformed using the $\boldsymbol{T}$ matrix, i.e.

$$\begin{aligned} \boldsymbol{p} &= \boldsymbol{x}_1 \times \boldsymbol{x}_2 \\ \boldsymbol{p}' &= \boldsymbol{x}_1' \times \boldsymbol{x}_2' = \boldsymbol{T}\boldsymbol{x}_1 \times \boldsymbol{T}\boldsymbol{x}_2 \\ &= \boldsymbol{Q}(\boldsymbol{x}_1 \times \boldsymbol{x}_2) \end{aligned} \quad (26)$$

The matrix $\boldsymbol{Q}$ is defined [4], [10] as

$$\boldsymbol{Q} = det(\boldsymbol{T})\ (\boldsymbol{T}^{-1})^T \quad (27)$$

Because $\boldsymbol{p}'$ are coefficients of an implicit equation we can simply write

$$\boldsymbol{p}' = det(\boldsymbol{T})(\boldsymbol{T}^{-1})^T(\boldsymbol{x}_1 \times \boldsymbol{x}_2) \quad (28)$$

As the implicit form is used, coefficients of a line can be multiplied by any non-zero constant and the line will be same. Therefore

$$\boldsymbol{p}' \triangleq (\boldsymbol{T}^{-1})^T(\boldsymbol{x}_1 \times \boldsymbol{x}_2) \quad (29)$$

where $\triangleq$ means protectively equal. Similarly for a plane $\boldsymbol{\rho}$

$$\boldsymbol{\rho}' \triangleq (\boldsymbol{T}^{-1})^T(\boldsymbol{x}_1 \times \boldsymbol{x}_2 \times \boldsymbol{x}_3) \quad (30)$$

It means that we can correctly manipulate with lines and planes, now.

D. Bounding Volume

Let us assume that the torus lies in the $x - z$ plane, i.e. the y-axis is its rotational axis. Bounding volume, defined in [1], is based on an idea that torus is bounded by an intersection of a sphere and two half-spaces, Fig.2.

The radius of the enclosing sphere is given as

$$R_1 = R + r \quad (31)$$

The test computes intersection a line with a sphere if such intersections t_{min} and t_{min} exist. In this case, the line does not intersect the torus if the following condition is valid [1]

$$\begin{gathered} y_{min} = y_A + s_y t_{min} \\ y_{max} = y_A + s_y t_{max} \\ (y_{min} > r\ \boldsymbol{and}\ y_{max} > r) \\ \boldsymbol{or}\ (y_{min} < r\ \boldsymbol{and}\ y_{max} < r) \end{gathered} \quad (32)$$

It can be seen that the test does not eliminate the cases when a line:

- is passing the "hole inside of the torus" without touching or intersecting the torus – line p_C
- nearly touches the torus – line p_D – but there is a small probability

It should be noted that the Fig.2 represents a general situation in E^3.

E. Torus Transformation

So far we have dealt with a general situation expecting that the torus is in its basic position, i.e. it lies in the $x - z$ plane and the y axis is the rotational axis. In the case of torus general position the following transformations can be used;

$$\boldsymbol{Q} = \begin{bmatrix} & \boldsymbol{u} & & 0 \\ & \boldsymbol{n} & & 0 \\ & \boldsymbol{u} \times \boldsymbol{n} & & 0 \\ -c_x & -c_y & -c_z & 1 \end{bmatrix} \quad (33)$$

where $\boldsymbol{u}$ defines x axis of the torus, $\boldsymbol{n}$ defines y axis of the torus, $\boldsymbol{u} \times \boldsymbol{n}$ is used to get an orthonormal basis, and $\boldsymbol{c}$ is the torus centre.

It can be seen that there are some interesting properties of the line-torus intersection problem, like

- torus rotational symmetry,
- if mirroring operation is used only one quadrant can be considered to solve the intersection problem.

F. Intersection Transformation

As a torus is rotationally invariant we can rotate the given line about y axis so that it lies in a plane $z = z_c$, i.e. in a plane parallel to $x - y$ plane. There is no significant computational expense as the transformation matrix is accumulated with the $\boldsymbol{Q}$ matrix. Now we can distinguish three fundamentally different cases according to the z_c value:

a) $0 \le z_c < R - r$: generally intersection of a line with two independent parts have to be considered, i.e. for $x > 0$ and $x < 0$ and due to convexity each part could have up to 2 intersections only (2 convex envelopes are generated),
b) $R - r \le z_c < R$: generally this case is more complex as the envelope has only one part, but it is not convex as it

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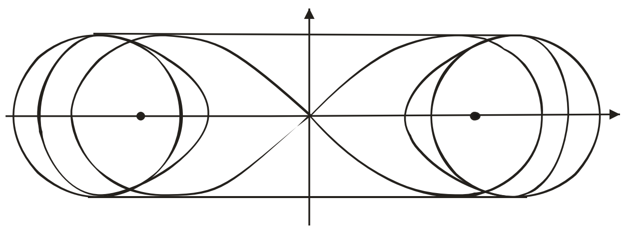
can have an inflexion point and 3 intersection points can be generated,

c) $R \leq z_c < R + r$: the simplest case as only one convex envelope is generated.

The above mentioned three cases differ significantly. Unfortunately the envelope is not convex in all the possible cases.

Fig. 3. Torus plane intersection for $z_c \leq R - r$



G. Vieta's Formula

Let us assume that $P(x)$ is a polynomial of degree n

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \quad (34)$$

Then according to the Vieta's formula [2] the roots satisfy equations

$$x_1 + x_1 + \cdots + x_1 = -\frac{a_{n-1}}{a_n}$$
$$(x_1x_2 + x_1x_3 + \cdots + x_1x_n) + (x_2x_3 + \cdots + x_2x_n) + \cdots + x_{n-1}x_n = \frac{a_{n-1}}{a_n} \quad (35)$$
$$\vdots$$
$$x_1x_2x_3 \ldots x_n = (-1)^n \frac{a_0}{a_n}$$

In the quadratic equation case

$$at^2 + bt + c = 0 \quad (36)$$

we obtain

$$t_1 + t_2 = -\frac{b}{a} \qquad t_1t_2 = \frac{c}{a} \quad (37)$$

These formulas are not well known and will be used latter on. In the following we will show different approaches to the line-torus intersection problem.

III. New Formulation

In the previous part we have presented the "traditional" approach to the line–torus intersection detection and computation. However, different equivalent formulations, which could lead to simpler and faster solutions, will be formulated in the following part. They can be briefly classified as follows:

- sphere is rotating about y axis (the envelope forms a torus) and intersection with the line in E^3 is computed directly,
- sphere is fixed on the x axis and intersection with the line rotating about y axis (intersection of a sphere and double cone) in E^3 is computed directly,
- sphere is rotating about y axis and intersection with the plane $z = z_c$ in E^3, on which the given line lies, results into circles in this plane, i.e. circles in E^2, forming an envelope of all those circles. An intersection of the envelope of all circles and the line is computed in E^2.

Let us explore the first possible formulation more in detail, now.

A. Sphere Rotation – Intersection in E^3

Let us consider a situation in which the torus and line are in the same relative position, but using the above mentioned geometric transformations, the torus is in its basic position, i.e. in the $x - z$ plane.

A torus can be represented as a union all spheres with a radius r rotating about y axis in the $x - z$ plane with a radius R. It means that the torus can be defined as a union, i.e. an envelope, of all rotating spheres about y axis as

$$\bigcup_{\varphi \in <0,2\pi)} (\boldsymbol{x} - \boldsymbol{x}_s(\varphi))^T(\boldsymbol{x} - \boldsymbol{x}_s(\varphi)) - r^2 = 0 \quad (38)$$

where

$$\boldsymbol{x}_s(\varphi) = [R cos\varphi, 0, R sin\varphi]^T \quad \varphi \in <0, 2\pi) \quad (39)$$

and the line in E^3 is defined as

$$\boldsymbol{x}(t) = \boldsymbol{x}_A + \boldsymbol{s}\, t \quad (40)$$

The line-torus intersection problem can be transformed to generalized line clipping problem, when a line is clipped by an envelope of all rotating K spheres which forms the T torus, i.e.

$$\mathrm{T} = \bigcup_{\varphi \in <0,2\pi)} \mathrm{K}(\varphi, R, r) \quad (41)$$

where R and r are given constants of the torus.

Due to the rotational symmetry about the y axis, the torus and the line can be rotated about y axis so that the line will lie in a plane parallel to the $x - y$ plane.

The directional vector $\boldsymbol{s}$ of the line is defined as

$$\boldsymbol{s} = [s_x, s_y, 0]^T \quad (42)$$

and the line lies in a plane parallel to the $x - y$ plane, i.e. ρ: $z = z_c$.

Now the problem of a line-torus intersection problem is transformed to generalized clipping problem in E^2 actually, when a line is clipped by the envelope.

A line is given in the case of E^3 as

$$\boldsymbol{x}(t) = \boldsymbol{x}_A + \boldsymbol{s}\, t \quad (43)$$

and a sphere

$$(\boldsymbol{x} - \boldsymbol{x}_s(\varphi))^T(\boldsymbol{x} - \boldsymbol{x}_s(\varphi)) - r^2 = 0 \quad (44)$$

substituting we get

$$[\boldsymbol{s}t - \boldsymbol{\xi}(\varphi)]^T[\boldsymbol{s}t - \boldsymbol{\xi}(\varphi)] - r^2 = 0 \qquad \boldsymbol{\xi}(\varphi) = \boldsymbol{x}_A - \boldsymbol{x}_s(\varphi) \quad (45)$$

i.e.

$$\boldsymbol{s}^T\boldsymbol{s}\, t^2 - 2\boldsymbol{s}^T\boldsymbol{\xi}(\varphi)t + \boldsymbol{\xi}^T(\varphi)\boldsymbol{\xi}(\varphi) = 0 \quad (46)$$

where

$$\boldsymbol{s}^T\boldsymbol{\xi}(\varphi) = \boldsymbol{s}^T\boldsymbol{x}_A - \boldsymbol{s}^T\boldsymbol{x}_s(\varphi)$$
$$\boldsymbol{\xi}^T(\varphi)\boldsymbol{\xi}(\varphi) = \boldsymbol{x}_A^T\boldsymbol{x}_A - 2\boldsymbol{x}_A^T\boldsymbol{x}_s(\varphi) + \boldsymbol{x}_s^T(\varphi)\boldsymbol{x}_s(\varphi) \quad (47)$$

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For normalized directional vector $\boldsymbol{s}$, i.e. $\|\boldsymbol{s}\| = 1$, resp. $\boldsymbol{s}^T\boldsymbol{s} = 1$, we get a quadratic equation parameterized by φ as follows

$$at^2 + bt + c = 0$$
$$a = 1 \quad b = -2\boldsymbol{s}^T\boldsymbol{\xi}(\varphi) \quad c = \boldsymbol{\xi}^T(\varphi)\boldsymbol{\xi}(\varphi) \tag{48}$$

If the Vieta's formula is used we get the following equivalent equations

$$t_1 + t_2 = 2\boldsymbol{s}^T\boldsymbol{\xi}(\varphi)$$
$$t_1 t_2 = \boldsymbol{\xi}^T(\varphi)\boldsymbol{\xi}(\varphi) \tag{49}$$

where

$$\boldsymbol{\xi}(\varphi) = \boldsymbol{x}_A - \boldsymbol{x}_s(\varphi) = \begin{bmatrix} x_A \\ y_A \\ z_A \end{bmatrix} - R\begin{bmatrix} cos\varphi \\ 0 \\ sin\varphi \end{bmatrix} \tag{50}$$

Re-parameterization of the line so that $y_A = 0$ leads to additional simplification.

$$\boldsymbol{\xi}(\varphi) = \boldsymbol{x}_A - \boldsymbol{x}_s(\varphi)$$
$$= \begin{bmatrix} x_A \\ 0 \\ z_A \end{bmatrix} - R\begin{bmatrix} cos\varphi \\ 0 \\ sin\varphi \end{bmatrix} = \begin{bmatrix} x_A - Rcos\varphi \\ 0 \\ z_A - Rsin\varphi \end{bmatrix} \tag{51}$$

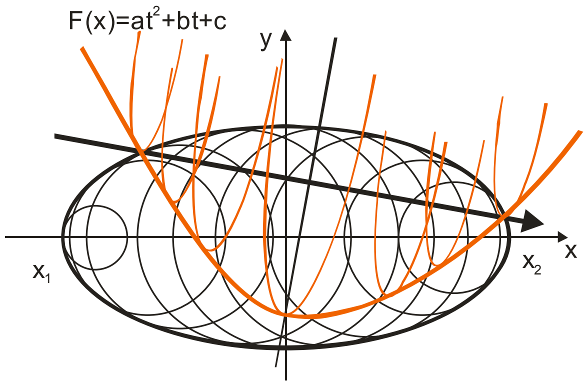
Let us consider a function $F(t)$

$$F(t) = t^2 + bt + c \tag{52}$$

then $F(t) = 0$ represents points of the line and the sphere $\mathrm{K}(\varphi, R, r)$, i.e. at the position given by the angle φ. A minimum value of the $F(t)$ function for the given φ is determined as

$$t_{extrem}(\varphi) = \frac{t_1 + t_2}{2} = \boldsymbol{s}^T\boldsymbol{\xi}(\varphi)$$
$$= [s_x, s_y, 0]\left(\begin{bmatrix} x_A - Rcos\varphi \\ 0 \\ z_A - Rsin\varphi \end{bmatrix}\right) \tag{53}$$
$$= s_x(x_A + Rcos\varphi)$$

Fig. 4. Rotating sphere-plane intersection and its envelope



The points x_1 and x_2, resp. parameters t_1 and t_2, can be determined an intersection of a plane on which the given line lies and $z = z_c$, Fig.5.

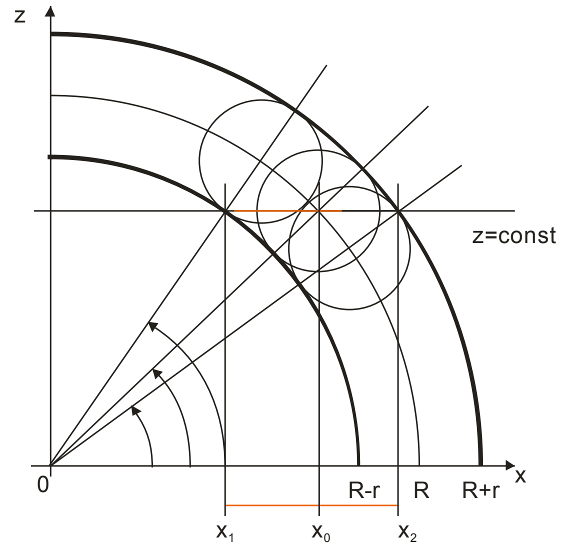
If $F(t_{extrem}(\varphi)) \leq 0$ then $< t_1, t_{extrem} >$ and $< t_{extrem}, t_2 >$ are intervals for iterative solution.

B. New Bounding Volume

The "standard" bounding volume [1] is based on a sphere in E[3] and an intersection of two half spaces, Fig.2. As the line lies in the $x - y$ plane for $z = z_c$ we can distinguish following fundamental cases:

- ad a) we can use mirroring operations and solve the intersection in one quadrant only twice for non-mirrored and for mirrored cases as there might be two tuples of intersections,
- ad b) situation is complex as the envelope has an inflection point so there might be three intersections in one quadrant,
- ad c) this case is similar to the previous but only two intersection points might occur.

Fig. 5. Rotating spheres



However if many lines-torus intersections computation are needed, like in the ray tracing rendering technique, the more precise bounding volume is needed to increase the efficiency of computation. The "standard" bounding volume works fine for the case ad b). On the other hand it can be seen that

- in the case ad a), i.e. when a line passes through the torus, i.e. through a "hole" and does not intersect the torus, detailed computation has to be made, that is computationally expensive.
- in the case ad c), i.e. when a line intersects the torus in its "outer part", i.e. $R \leq z_c < R + r$ the distance between two planes can be smaller than $2r$.

Let us explore the first case as it leads to higher efficiency.

Fig. 6. Rotating spheres

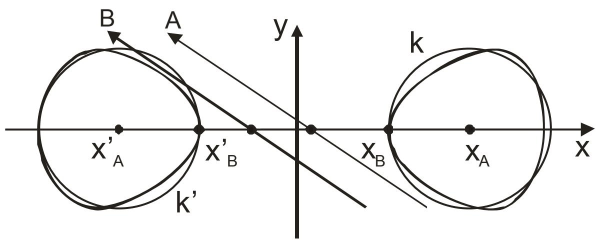


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Fig. 7. Torus intersection with a plane on which the line lies – case ad b)

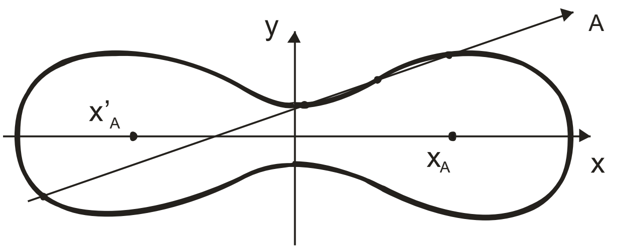


Fig.6 presents an intersection plane-torus for $0 < z_c < R - r$. It can be seen that a k circle (as we are in E^2), with the center at x_A with the radius r forms bounding surfaces together with the mirrored k' circle by y axis. The x_A center of the circle is defined as follows:

$$x_B = (R-r)cos\varphi \qquad x_A = x_B + r = x_B + (R-r)cos\varphi \tag{38}$$

where

$$sin\varphi = \frac{z_c}{R} \tag{39}$$

The test for the ad a) case can be formulated as: if the line intersects the x axis in the interval (x'_B, x_B) and does not intersect the circle k nor the circle k' , then the line does not intersect the given torus. Fig.6 presents two lines, in the case A, the line is not considered for intersection computation with torus, while in the cases B, the detailed intersection test/computation has to be made.

The test for the ad b) test remains as the original, Fig.7, as up to 3 intersections can occur in one quadrant as there is a point of inflexion.

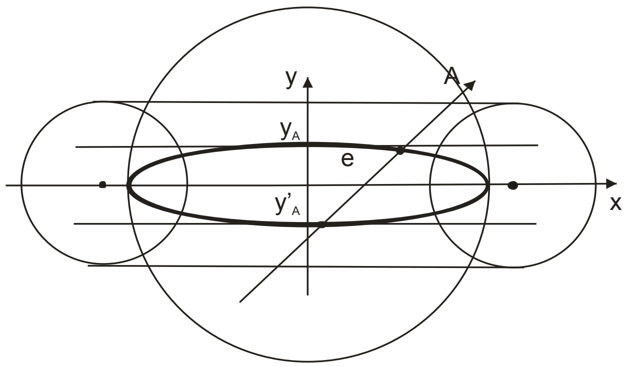
In the case ad c), i.e. $R \leq z_c < R + r$, there are only 2 intersection points possible. It can be seen that the distance between two planes, given by y_A and y'_A values is now smaller than the original distance $2r$. It can be seen that the new distance is given as

$$d^2 = r^2 - (z_c - r)^2 \tag{40}$$

I. Conclusion

A new approach to line-torus intersection problem has been presented and a new bounding object, actually a circle in E^2, for the line-torus intersection has been developed and described. The new bounding object increases line-torus intersection computation efficiency significantly as it also detects the cases when a line or ray is passing a "hole" of the torus. The efficiency of the new torus bounding test grows with the ratio $\nu = R/r$.

Figure 8: Torus bounding for $R \leq z_c < R + r$



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