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Projective Geometry and Duality for Graphics, Games and Visualization



An Introductory Course

Singapore

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Plzen (Pilsen) City

Plzen is an old city [first records of Plzen castle 976] city of culture, industry, and brewery.

City, where today's beer fermentation process was invented
that is why today's beers are called Pilsner - world wide

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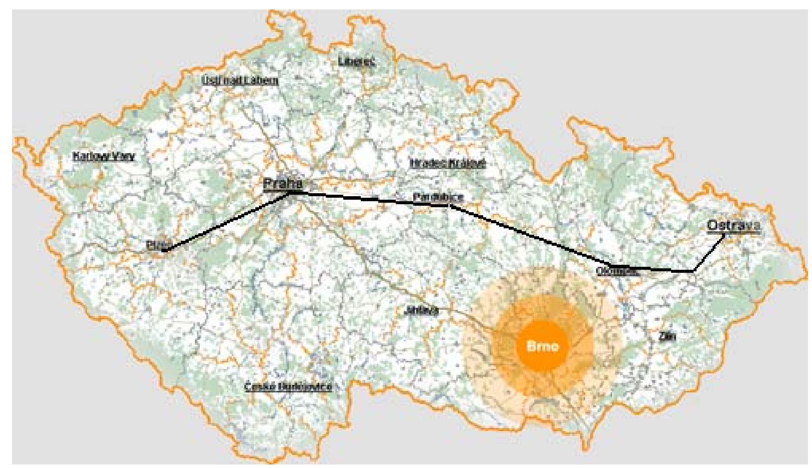
Ostrava City

Ostrava is

- an industrial city of coal mining & iron melting
- 3rd largest city



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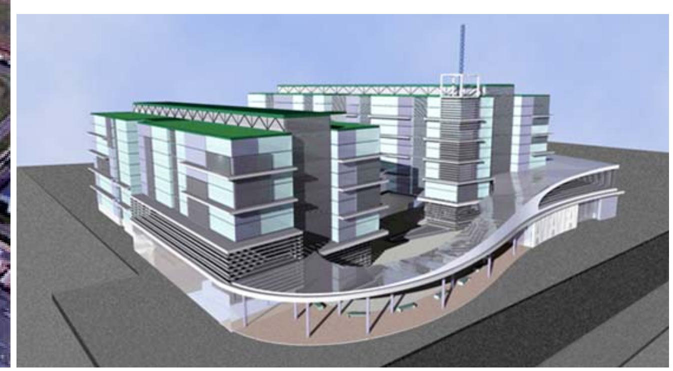
University of West Bohemia 17530 students + 987 PhD students

Computer Science and Engineering **Mathematics** (+ Geomatics)
Physics **Cybernetics** **Mechanics** (Computational)

- Over **50%** of income from research and application projects
- NTIS project (investment of 64 mil. EUR)
- 2^{nd} in the ranking of Czech technical / informatics faculties 2009, 2012



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"Real science" in the XXI century

Courtesy of Czech Film, Barrandov

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An overview

- Precision and robustness
- Euclidean space and projective extension
- Principle of duality and its applications
- Geometric computation in the projective space
- Intersection of two planes in E3 with additional constrains
- Barycentric coordinates and intersections
- Interpolation and intersection algorithms
- Implementation aspects and GPU
- Conclusion and summary

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Numerical systems

- Binary system is used nearly exclusively
- Octal & hexadecimal representation is used
- If we would be direct descendants of tetrapods – we would have a great advantage – "simple counting in hexadecimal system"

	Name	Base	Digits	E min	E max
BINARY					
B 16	Half	2	10+1	−14	15
B 32	Single	2	23+1	−126	127
B 64	Double	2	52+1	−1022	1023
B 128	Quad	2	112+1	−16382	16383
DECIMAL					
D 32		10	7	−95	96
D 64		10	16	−383	384
D 128		10	34	−6143	6144

IEEE 758-2008 standard

Courtesy Clive "Max" Maxfield and Alvin Brown

The first tetrapods had eight fingers on each hand

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Mathematically perfect algorithms fail due to instability

Main issues

- stability, robustness of algorithms
- acceptable speed
- linear speedup – results depends on HW, CPU …. parameters !

Numerical stability

- limited precision of float / double
- tests A ? B with floats

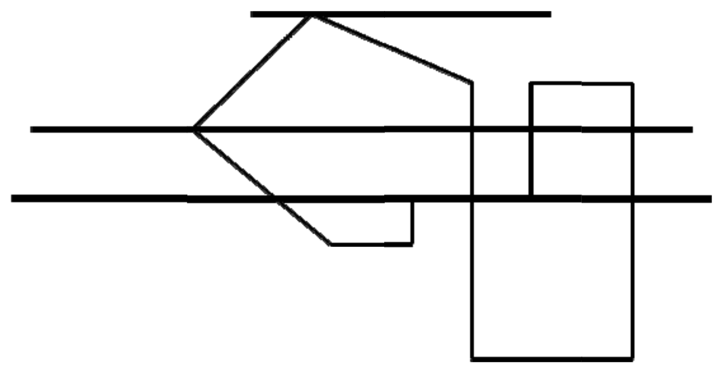
  if A = B **then** ….. **else** ….. ; **if** A = 0 **then** ….. **else** ….
  should be forbidden in programming languages

- division operation should be removed or postponed to the last moment if possible - "blue screens", system resets

Typical examples of instability

- intersection of 2 lines in E^3
- point lies on a line in E^2 or a plane in E^3

  $Ax + By + C = 0$ or $Ax + By + Cz + D = 0$
- detection if a line intersects a polygon, touches a vertex or passes through

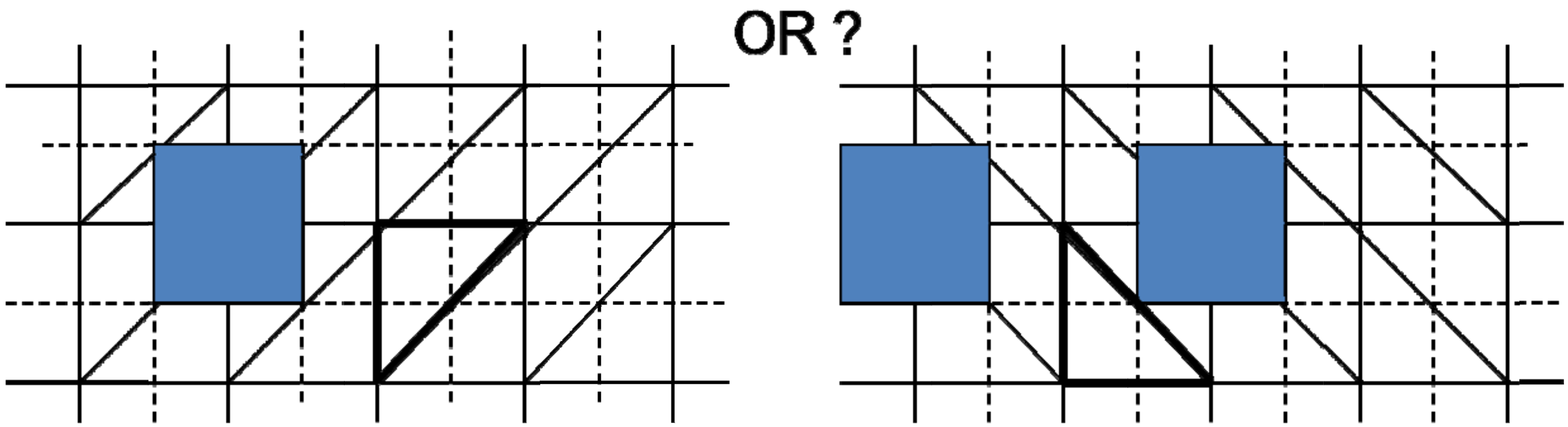


Typical problem

```
double x = -1; double p = ….;
while ( x < +1)
{    if (x == p) Console.Out.WriteLine(" *** ")
     x += p;
}
/*    if p = 0.1 then no output,   if p = 0.25 then expected output */
```

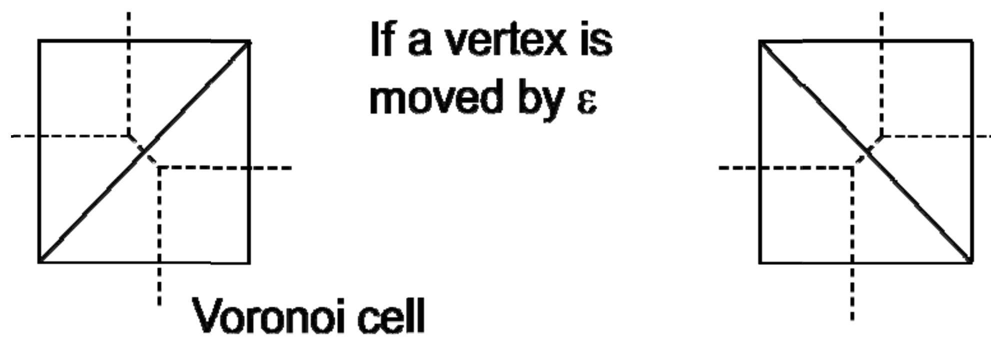
Delaunay triangulation & Voronoi diagram

Point inside of a circle given by three points – problems with meshing points in regular rectangular grid.



It can be seen that the DT & VD is very sensitive to a point position change

?? ROBUSTNESS ??



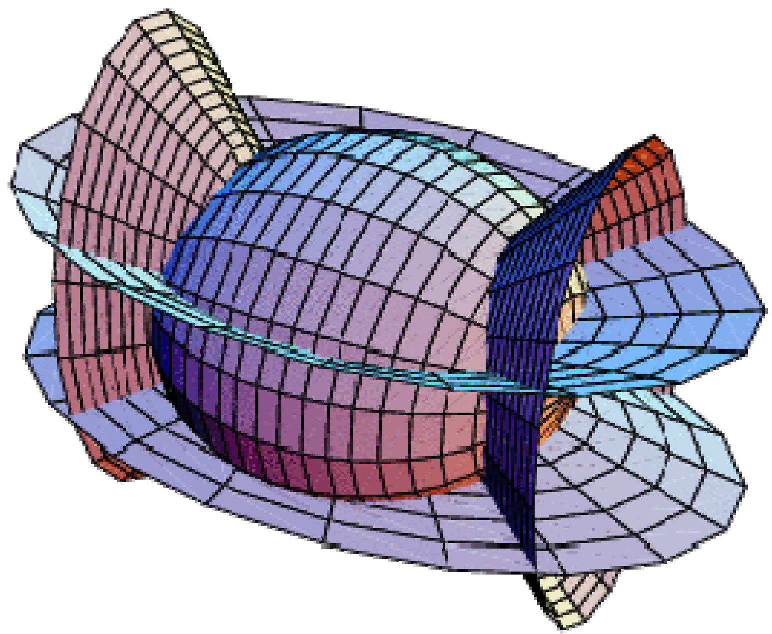
Vectors and Points in Geometry

Vectors – movable, no fixed position

- Points – no size, position fixed in the GIVEN coordinate system

Coordinate systems:

- Cartesian – left / right handed
  right handed system is used
- Polar
- Spherical
- and many others, e.g. Confocal Ellipsoidal Coordinates (http://mathworld.wolfram.com/ConfocalEllipsoidalCoordinates.html)



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Floating point

- Not all numbers are represented correctly
- Logarithmic arithmetic
- Continuous fractions
- Interval arithmetic

$$\pi = \cfrac{4}{1 + \cfrac{1^2}{3 + \cfrac{2^2}{5 + \cfrac{3^2}{\dots}}}}$$

$$\pi = [3; 7,15,1,292,1,1,1,2,1,3,1 \dots]$$

Generally NOT valid identities due to limited precision

- $cos^2\alpha + cos^2\beta = 1$
  $[\ \alpha + \beta = \pi\ ]$
- $\mathrm{x}^2 - \mathrm{y}^2 = (x - y)(x + y)$

x + y = [a + c, b + d]	x = [a , b]
x - y = [a - d, b - c]	y = [c , d]

x × y = [min(ac, ad, bc, bd), max(ac, ad, bc, bd)]

x / y = [min(a/c, a/d, b/c, b/d),

max(a/c, a/d, b/c, b/d)] if y $\neq$ 0

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Statements like

if <float> = <float> **then ….** or **if** <float> ≠ <float> **then ….**

should not be allowed in languages

Quadratic equation

$at^2 + bt + c = 0$ usually solved as $t_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

If $b^2 \gg 4ac$ then

$$q = -(b + sign(b)\sqrt{b^2 - 4ac})/2$$

$t_1 = {}^{q}/_{a}$ $t_2 = {}^{c}/_{a}$

to get more reliable results.

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Function value computation at $x = 77617$, $y = 33096$

$$f(x,y) = 333.75y^6 + x^2(11x^2y^2 - y^6 - 121y^4 - 2) + 5.5y^8 + x/(2y)$$

$f = 6.33835\ 10^{29}$ single precision

$f = 1{,}1726039400532$ double precision

$f = 1{,}172603940053178631858834904520183$ extended precision

The correct result is "somewhere" in the interval of

$$[-0{,}827396059946821368141165095479816292\mathbf{005},$$
$$-0{,}827396059946821368141165095479816291\mathbf{986}]$$

Exact solution

$$f(x,y) = -2 + \frac{x}{2y} = -\frac{54767}{66192}$$

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Summation is one of often used computations.

$$\sum_{i=1}^{10^3} 10^{-3} = 0.999990701675415$$

or

$$\sum_{i=1}^{10^4} 10^{-4} = 1.000053524971008$$

The result should be only one.

The correctness in summation is very important in power series computations.
!!!!ORDER of summation

$$\sum_{n=1}^{10^6} \frac{1}{n} = 14.357357 \qquad \sum_{n=10^6}^{1} \frac{1}{n} = 14.392651$$

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Recursion

Towers of Hanoi

```
MOVE (A, C, n);
{   MOVE (A, B, n-1);
    MOVE (A, C, 1);
    MOVE (B, C, n-1)
}  # MOVE (from, to, number) #
```

Ackermann function

$$A(m,n) = \begin{cases} n+1 & if\ m=0 \\ A(m-1,1) & if\ M>0\ and\ n=0 \\ A\big(m-1, A(m,n-1)\big) & if\ m>0\ and\ N>0 \end{cases}$$

The value of the function grows very fast [WiKi] as

$$A(4,4) = 2^{2^{2^{65536}}} = 2^{2^{10^{197296}}}$$

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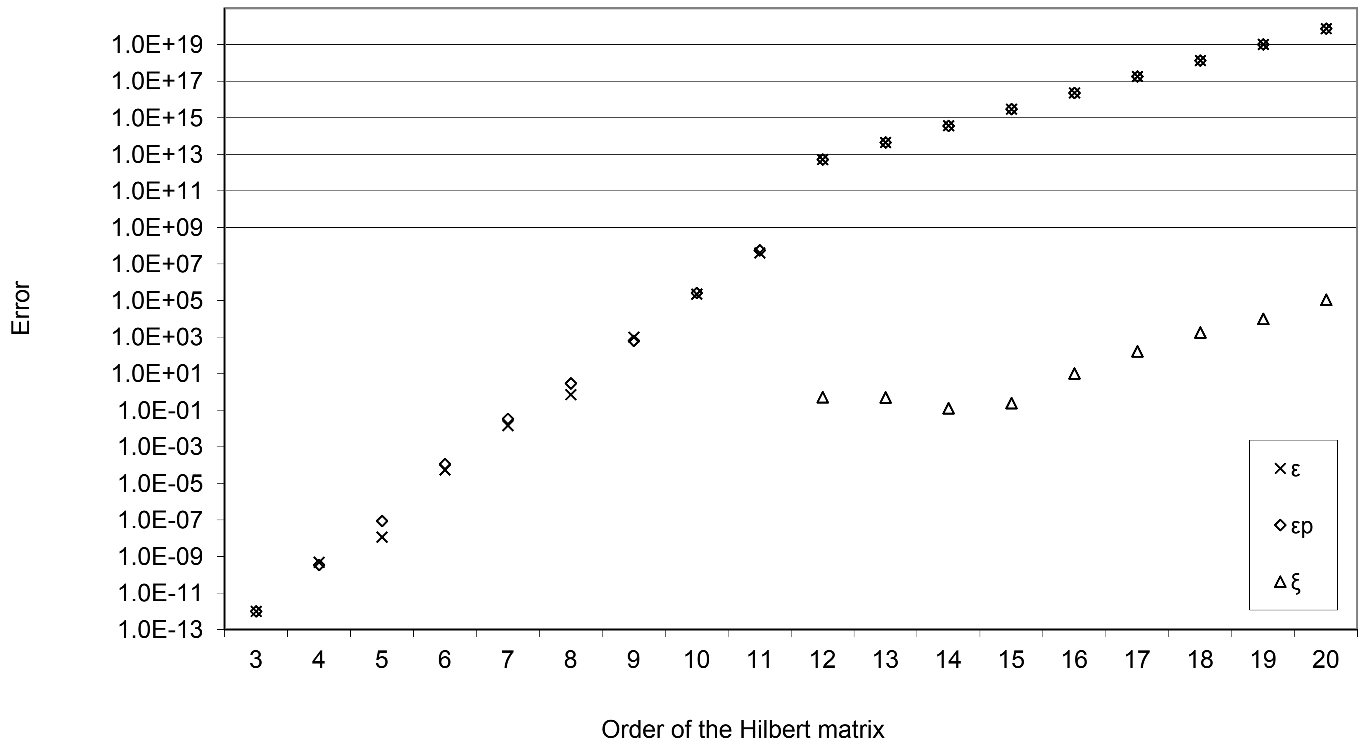
Numerical computations $\boldsymbol{Ax = b}$ $\boldsymbol{x = A^{-1}b}$

Hilbert's Matrix $H_{ij} = \frac{1}{i+j-1}$

$$H_{ij}^{-1} = (-1)^{i+j}(i+j-1)\binom{n+i-1}{n-j}\binom{n+j-1}{n-i}\binom{i+j-2}{i-1}^2$$



Mathematical "forms"

There are several "forms":

Implicit $F(x,y,z)=0$ or $F(x)=0$ or $\boldsymbol{F}(\boldsymbol{x})=\boldsymbol{0}$ (system of equations)

There is no orientation, e.g.

- if $F(\boldsymbol{x})=0$ is a iso-curve there is no hint how to find another point of this curve, resp. a line segment approximating the curve => tracing algorithms
- if $F(\boldsymbol{x})=0$ is a iso-surface there is no hint how to find another point of this surface => iso-surface extraction algorithms

Parametrical $\boldsymbol{x}=\boldsymbol{x}(u)$ $\boldsymbol{x}=\boldsymbol{x}(u,v)$

Points of a curve are "ORDERED" according to a parameter u, resp. u,v

Explicit $z=f(x)$ $z=f(x,y)$

For the given values x, resp. x,y we get function value z

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Implicit form

- Is used for separation - for detection if a point is inside or outside, e.g. a half-plane or a circle etc.

- There is always a question how to compute some formula

- Complexity of computations × precision of computation

Compiler optimization is DANGEROUS in general

$$\begin{vmatrix} A_x & A_y & A_x^2 + A_y^2 & 1 \\ B_x & B_y & B_x^2 + B_y^2 & 1 \\ C_x & C_y & C_x^2 + C_y^2 & 1 \\ D_x & D_y & D_x^2 + D_y^2 & 1 \end{vmatrix} = \begin{vmatrix} A_x - D_x, & A_y - D_y, & (A_x^2 - D_x^2) + (A_y^2 - D_y^2) \\ B_x - D_x, & B_y - D_y, & (B_x^2 - D_x^2) + (B_y^2 - D_y^2) \\ C_x - D_x, & C_y - D_y, & (C_x^2 - D_x^2) + (C_y^2 - D_y^2) \end{vmatrix} > 0$$

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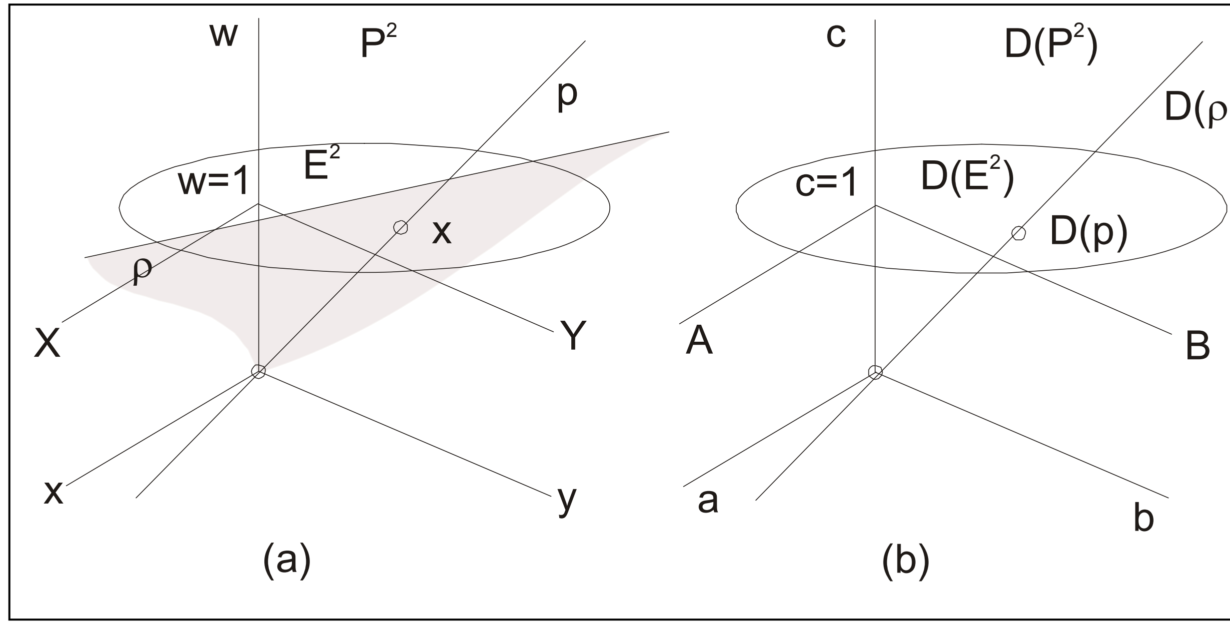
Projective Space

$\boldsymbol{X} = [X, Y]^T \quad \boldsymbol{X} \in E^2$

$\boldsymbol{x} = [x, y: w]^T \ \boldsymbol{x} \in P^2$



Conversion:

$X = x / w \qquad Y = y / w$

$\& \ w \neq 0$

If $w = 0$ then $\boldsymbol{x}$ represents "an ideal point" - a point in infinity, i.e. it is a directional vector.

The Euclidean space E^2 is represented as a plane $w = 1$.

Points and vectors

- Vectors **are "freely movable"** – not having a fixed position

$$\boldsymbol{a}_1 = [x_1, y_1 : 0]^T$$

- Points are **not "freely movable"** – they are fixed to an origin of the current coordinate system

$$\boldsymbol{x}_1 = [x_1, y_1 : w_1]^T \quad \text{and} \quad \boldsymbol{x}_2 = [x_2, y_2 : w_2]^T$$

usually in textbooks $w_1 = w_2 = 1$

A vector $\boldsymbol{A} = \boldsymbol{X}_2 - \boldsymbol{X}_1$ in the Euclidean coordinate system - **CORRECT**

Horrible "construction" DO NOT USE IT – IT IS TOTALLY WRONG

$$\boldsymbol{a} = \boldsymbol{x}_2 - \boldsymbol{x}_1 = [x_2 - x_1, y_2 - y_1 : w_2 - w_1]^T = [x_2 - x_1, y_2 - y_1 : 1 - 1]^T$$
$$= [x_2 - x_1, y_2 - y_1 : 0]^T$$

This was presented as "How a vector" is constructed in the projective space P^k in a textbook!! **WRONG, WRONG, WRONG**

This construction has been found in SW as well!!

Points and vectors

A vector given by two points in the projective space

$$\boldsymbol{a} = \boldsymbol{x}_2 - \boldsymbol{x}_1 = [w_1 x_2 - w_2 x_1, w_1 y_2 - w_2 y_1 : w_1\, w_2]^T$$

This is the **CORRECT SOLUTION**, but what is the interpretation?

A "difference" of coordinates of two points is a vector in the mathematical meaning and $w_1\, w_2$ is a "scaling" factor actually

In the projective representation (if the vector length matters)

$$\boldsymbol{a} = \boldsymbol{x}_2 - \boldsymbol{x}_1 = [w_1 x_2 - w_2 x_1, w_1 y_2 - w_2 y_1 : w_1\, w_2]^T$$
$$\triangleq \left[\frac{w_1 x_2 - w_2 x_1}{w_1\, w_2}, \frac{w_1 y_2 - w_2 y_1}{w_1\, w_2} : 0\right]^T$$

We have to strictly distinguish if we are working with points, i.e. vector as a data structure represents the coordinates, or with a vector in the mathematical meaning stored in a vector data structure.

VECTORS x FRAMES

Duality

For simplicity, let us consider a line p defined as:

$aX + bY + c = 0$

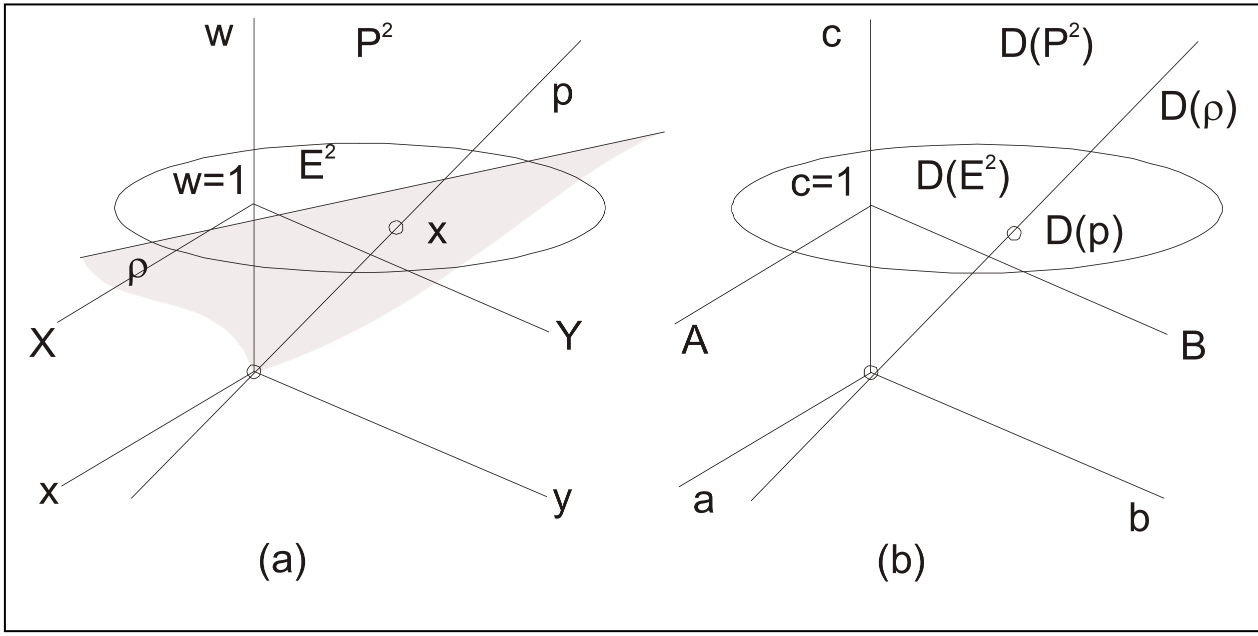
We can multiply it by $w \neq 0$ and we get:

$$ax + by + cw = 0$$

i.e. $\boldsymbol{p}^T\boldsymbol{x} = 0$

$\boldsymbol{p} = [\, a, b : c]^T$

$\boldsymbol{x} = [\, x, y : w]^T = [\, wX, wY : w]^T$



A line $p \in E^2$ is actually a plane in the projective space P^2 (point $[0,0:0]^T$ excluded)

Duality

From the mathematical notation $\boldsymbol{p}^{\boldsymbol{T}}\boldsymbol{x} = 0$

we cannot distinguish whether ***p*** is a line and ***x*** is a point or vice versa in the case of P^2. It means that

- a *point* and a *line* **are dual** in the case of P^2, and
- a *point* and a *plane* **are dual** in the case of P^3.

The principle of duality in P^2 states that:

Any theorem remains true when we interchange the words “point” and “line”, “lie on” and “pass through”, “join” and “intersection”, “collinear” and “concurrent” and so on.

Once the theorem has been established, the dual theorem is obtained as described above.

This helps a lot to solve some geometrical problems.

Examples of dual objects and operators

	Primitive	Dual primitive
E^2	Point Line	Line Point
E^3	Point Plane	Plane Point

Operator	Dual operator
Join Intersect	Intersect Join

***Computational sequence for a problem is the same for a dual problem*.**

Intersection of two lines

Let two lines p_1 and p_2 are given by

$$p_1 = [a_1, b_1 : c_1]^T \qquad \text{and} \qquad p_2 = [a_2, b_2 : \ c_2]^T$$

We have to solve a system of linear equations $\boldsymbol{Ax = b}$

$$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} -c_1 \\ -c_2 \end{bmatrix} \ *$$

Then well known formula is used

$$X = \frac{Det_X}{Det} = \frac{det \begin{bmatrix} q_1 & b_1 \\ q_2 & b_2 \end{bmatrix}}{det \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}} \qquad Y = \frac{Det_Y}{Det} = \frac{det \begin{bmatrix} a_1 & q_1 \\ a_2 & q_2 \end{bmatrix}}{det \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}}$$

But what if Det is small?

Usually a sequence like $\boldsymbol{if}\ abs(\det(..)) \le eps\ \boldsymbol{then}$ is used. What is eps?

Note * usually a line is in its explicit form as $ax + by = q$ instead of $+by + c = 0$, i.e. the implicit form

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How a line is given by two X_1 and X_2 points?

We have to solve a homogeneous system of linear equations

$$\begin{bmatrix} X_1 & Y_1 & 1 \\ X_2 & Y_2 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

i.e. $\boldsymbol{Ax} = \mathbf{0}$. It means that there is ***one parametric set*** of solutions!

NOW

Computation of

- an intersection of two lines is given as $\boldsymbol{Ax} = \boldsymbol{b}$
- a line given by two points is given as $\boldsymbol{Ax} = \mathbf{0}$

Different schemes for computation BUT

Those problems are DUAL, why algorithms are different??

Definition

The cross product of the two vectors

$\mathbf{x}_1 = [x_1, y_1 : w_1]^T$ and $\mathbf{x}_2 = [x_2, y_2 : w_2]^T$

is defined as:

$$\boldsymbol{x}_1 \times \boldsymbol{x}_2 = det \begin{bmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ x_1 & y_1 & w_1 \\ x_2 & y_2 & w_2 \end{bmatrix}$$

where: $\mathbf{i} = [1,0:0]^T$, $\mathbf{j} = [0,1:0]^T$, $\mathbf{k} = [0,0:1]^T$

Please, note that homogeneous coordinates are used.

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Theorem

Let two points $\mathbf{x}_1$ and $\mathbf{x}_2$ be given in the projective space. Then the coefficients of the ***p*** line, which is defined by those two points, are determined as the cross product of their homogeneous coordinates

$$\boldsymbol{p} = \boldsymbol{x}_1 \times \boldsymbol{x}_2 = [a, b : c]^T$$

Proof

Let the line $\boldsymbol{p} \in E^2$ be defined in homogeneous coordinates as (coefficient d is used intentionally to have the same symbol representing a “distance” of the element from the origin for lines and planes)

$$ax + by + cw = 0$$

We are actually looking for a solution to the following equations:

$$\boldsymbol{p}^T \boldsymbol{x}_1 = 0 \qquad\qquad \boldsymbol{p}^T \boldsymbol{x}_2 = 0$$

where: $\mathbf{p} = [a, b : c]^T$

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It means that any point $\boldsymbol{x}$ that lies on the p line must satisfy both the equation, i.e. $\boldsymbol{p}^T\boldsymbol{x}_1 = 0$ $\boldsymbol{p}^T\boldsymbol{x}_2 = 0$ and the equation $\boldsymbol{p}^{\boldsymbol{T}}\boldsymbol{x} = 0$ in other words the $\boldsymbol{p}$ vector is defined as

$$\boldsymbol{p} = \boldsymbol{x}_1 \times \boldsymbol{x}_2 = det\begin{bmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ x_1 & y_1 & w_1 \\ x_2 & y_2 & w_2 \end{bmatrix}$$

We can write

$$(\boldsymbol{x}_1 \times \boldsymbol{x}_2)^{\boldsymbol{T}}\boldsymbol{x} = 0 \qquad \text{i.e.} \quad det\begin{bmatrix} x & y & w \\ x_1 & y_1 & w_1 \\ x_2 & y_2 & w_2 \end{bmatrix} = 0$$

Note that **cross** product and **dot** product are instructions in Cg/HLSL on GPU

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Evaluating the determinant $det \begin{bmatrix} a & b & c \\ x_1 & y_1 & w_1 \\ x_2 & y_2 & w_2 \end{bmatrix} = 0$

we get the line coefficients of the line *p* as:

$$a = det \begin{bmatrix} y_1 & w_1 \\ y_2 & w_2 \end{bmatrix} \qquad b = -det \begin{bmatrix} x_1 & w_1 \\ x_2 & w_2 \end{bmatrix} \qquad c = det \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix}$$

Note:

1. A line $ax + by + c = 0$ is a one parametric set of coefficients

   $$\boldsymbol{p} = [a, b : c]^T$$

   From two values $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ we have to compute 3 values, coefficients a , b and c

2. For w = 1 we get the standard cross product formula and the cross product defines the *p* line, i.e. $\boldsymbol{p} = \boldsymbol{x}_1 \times \boldsymbol{x}_2$ where:

   $$\boldsymbol{p} = [a, b : c]^T$$

We have seen, in the Euclidean space, the computation of

- an intersection of two lines is given as $\boldsymbol{Ax} = \boldsymbol{b}$
- a line given by two points is given as $\boldsymbol{Ax} = \boldsymbol{0}$

If projective representation is used it is actually an application of the cross product.

Those problems are DUAL and algorithms are identical ??

Cross product is equivalent to a solution of a linear system of equations!

No division operations!

DUALITY APPLICATION

In the projective space P^2 points and lines are dual. Due to duality we can directly intersection of two lines as

$$\boldsymbol{x} = \boldsymbol{p}_1 \times \boldsymbol{p}_2 = det \begin{bmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{bmatrix} = [x, y: w]^T$$

If the lines are parallel or close to parallel, the homogeneous coordinate $w \to 0$ and users have to take a decision – so there is no sequence in the code like $\boldsymbol{if}\ abs(\det(..)) \le eps\ \boldsymbol{then}$ …in the procedure.

Generally computation can continue even if $w \to 0$ if projective space is used.

DISTANCE

Geometry is strongly connected with distances and their measurement, geometry education is strictly sticked to the Euclidean geometry, where the distance is measured as

$$d = \sqrt{(\Delta x)^2 + (\Delta y)^2} \quad \text{, resp.} \quad d = \sqrt{(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2} .$$

This concept is convenient for a solution of basic geometric problems, but in many cases it results into quite complicated formula and there is a severe question of stability and robustness in many cases.

The main objection against the projective representation is that there is no metric.

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The distance of two points can be easily computed as

$$dist = \sqrt{\xi^2 + \eta^2}/(w_1 w_2)$$

where: $\xi = w_1 x_2 - w_2 x_1$ $\eta = w_1 y_2 - w_2 y_1$

Also a distance of a point $\boldsymbol{x}_0$ from a line in E^2 can be computed as

$$dist = \frac{\boldsymbol{a}^T \boldsymbol{x}_0}{w_0 \sqrt{a^2 + b^2}}$$

where: $\boldsymbol{x_0} = [x_0, y_0 : w_0]^T$ $\boldsymbol{a} = [a, b : c]^T$

The extension to E^3/P^3 is simple and the distance of a point $\boldsymbol{x}_0$ from a plane in E^3 can be computed as

$$dist = \frac{\boldsymbol{a}^T \boldsymbol{x}_0}{w_0 \sqrt{a^2 + b^2 + c^2}}$$

where: $\boldsymbol{x_0} = [x_0, y_0, z_0 : w_0]^T$ $\boldsymbol{a} = [a, b, c : d]^T$.

In many cases we do not need actually a *distance*, e.g. for a decision which object is closer, and *distance*^{2} can be used instead, i.e. for the E^2 case

$$dist^2 = \frac{(\boldsymbol{a}^T \boldsymbol{x}_0)^2}{{w_0}^2(a^2 + b^2)} = \frac{(\boldsymbol{a}^T \boldsymbol{x}_0)^2}{{w_0}^2\, \boldsymbol{n}^T \boldsymbol{n}}$$

where: $\boldsymbol{a} = [a, b : c]^T = [\boldsymbol{n} : c]^T$ and the normal vector $\boldsymbol{n}$ is not normalized

If we are comparing distances of points from the given line *p* we can use "*pseudo-distance*" for comparisons

$$(pseudo\_dist)^2 = \frac{(\boldsymbol{a}^T \boldsymbol{x}_0)^2}{{w_0}^2}$$

Similarly for a plane ρ in the case of E^3

$$dist^2 = \frac{(\boldsymbol{a}^T \boldsymbol{x}_0)^2}{{w_0}^2(a^2 + b^2 + c^2)} = \frac{(\boldsymbol{a}^T \boldsymbol{x}_0)^2}{{w_0}^2\, \boldsymbol{n}^T \boldsymbol{n}}$$ and $$(pseudo\_dist)^2 = \frac{(\boldsymbol{a}^T \boldsymbol{x}_0)^2}{{w_0}^2}$$

where: $\boldsymbol{a} = [a, b, c : d]^T = [\boldsymbol{n} : d]^T$

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Computation in Projective Space

- Cross product definition
- A plane ***ρ*** is determined as a cross product of three given points

$$\mathbf{x}_1 \times \mathbf{x}_2 \times \mathbf{x}_3 = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} & \mathbf{l} \\ x_1 & y_1 & z_1 & w_1 \\ x_2 & y_2 & z_2 & w_2 \\ x_3 & y_3 & z_3 & w_3 \end{bmatrix}$$

Due to the duality

- An intersection point ***x*** of three planes is determined as a cross product of three given planes

$$\boldsymbol{\rho}_1 \times \boldsymbol{\rho}_2 \times \boldsymbol{\rho}_3 = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} & \mathbf{l} \\ a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{bmatrix}$$

Computation of generalized cross product is equivalent to a solution of a linear system of equations => no division operation!

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Geometric transformations with points

(note $X = {}^{x}/_{w}$, $Y = {}^{y}/_{w}$, $w \neq 0$):

Translation by a vector $(A, B) \triangleq [a, b: c]^T$, i.e. $A = a/c$, $B = b/c$, $c \neq 0$:

In the Euclidean space: $\boldsymbol{x}' = \boldsymbol{T}\boldsymbol{x}$

$$\begin{bmatrix} x' \\ y' \\ w \end{bmatrix} = \begin{bmatrix} 1 & 0 & A \\ 0 & 1 & B \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix} = \begin{bmatrix} x + Aw \\ y + Bw \\ w \end{bmatrix} \triangleq \begin{bmatrix} x/w + A \\ y/w + B \\ 1 \end{bmatrix} = \begin{bmatrix} X + A \\ Y + A \\ 1 \end{bmatrix}$$

In the projective space: $\boldsymbol{x}' = \boldsymbol{T}'\boldsymbol{x}$

$$\begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} = \begin{bmatrix} c & 0 & a \\ 0 & c & b \\ 0 & 0 & c \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix} = \begin{bmatrix} cx + aw \\ cy + bw \\ cw \end{bmatrix} \triangleq \begin{bmatrix} (cx + aw)/(cw) \\ (cy + bw)/(cw) \\ 1 \end{bmatrix} = \begin{bmatrix} {}^{x}/_{w} + {}^{a}/_{c} \\ {}^{y}/_{w} + {}^{b}/_{c} \\ 1 \end{bmatrix} = \begin{bmatrix} X + A \\ Y + B \\ 1 \end{bmatrix}$$

and $\det(\boldsymbol{T}') = c^3$. For $c = 1$ we get a standard formula

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & A \\ 0 & 1 & B \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

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Rotation by an angle $(cos\varphi, sin\varphi) = \left(\frac{a}{c}, \frac{b}{c}\right) \triangleq [a, b: c]^T$

In the Euclidean space: $\boldsymbol{x}' = \boldsymbol{R}\boldsymbol{x}$

$$\begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} = \begin{bmatrix} cos\varphi & -sin\varphi & 0 \\ sin\varphi & cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix} \triangleq \begin{bmatrix} cos\varphi & -sin\varphi & 0 \\ sin\varphi & cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ 1 \end{bmatrix}$$

In the projective space: $\boldsymbol{x}' = \boldsymbol{R}'\boldsymbol{x}$

$$\begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} = \begin{bmatrix} a & -b & 0 \\ b & a & 0 \\ 0 & 0 & c \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix} = \begin{bmatrix} ax - by \\ bx + ay \\ cw \end{bmatrix} \triangleq$$

$$\begin{bmatrix} (ax - by)/(cw) \\ (bx + ay)/(cw) \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{x}{w}\frac{a}{c} - \frac{y}{w}\frac{b}{c} \\ \frac{x}{w}\frac{b}{c} + \frac{y}{w}\frac{a}{c} \\ 1 \end{bmatrix} = \begin{bmatrix} Xcos\varphi - Ysin\varphi \\ Xsin\varphi + Ycos\varphi \\ 1 \end{bmatrix}$$

as $c^2 = (a^2 + b^2)$ by definition, $\det(\boldsymbol{R}') = (a^2 + b^2)c = c^3$

Scaling by a factor $(S_x, S_y) = (\frac{s_x}{w_s}, \frac{s_y}{w_s}) \triangleq [s_x, s_y : w_s]^T$

$$\boldsymbol{x}' = \boldsymbol{S}\boldsymbol{x} \qquad \begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} = \begin{bmatrix} S_x & 0 & 0 \\ 0 & S_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix}$$

$$\boldsymbol{x}' = \boldsymbol{S}'\boldsymbol{x} \qquad \begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & w_s \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix}$$

$$\det(\boldsymbol{S}') = s_x \, s_y \, w_s$$

It is necessary to note that the determinant of a transformation matrix $\boldsymbol{Q}$, i.e. matrices $\boldsymbol{T}', \boldsymbol{R}', \boldsymbol{S}'$, is $\det(\boldsymbol{Q}) \neq 1$ in general, but as the formulation is in the projective space, there is no need to “normalize” transformations to $\det(\boldsymbol{Q}) = 1$ even for rotation.

It can be seen that if the parameters of a geometric transformation are given in the homogeneous coordinates, no division operation is needed at all.

Transformation of lines and planes

	E^2	E^3
	$\boldsymbol{p} = \boldsymbol{x}_1 \times \boldsymbol{x}_2$	$\boldsymbol{\rho} = \boldsymbol{x}_1 \times \boldsymbol{x}_2 \times \boldsymbol{x}_3$
Dual problem	$\boldsymbol{x} = \boldsymbol{p}_1 \times \boldsymbol{p}_2$	$\boldsymbol{x} = \boldsymbol{\rho}_1 \times \boldsymbol{\rho}_2 \times \boldsymbol{\rho}_3$

In graphical applications position of points are changed by an interaction, i.e. $\boldsymbol{x}' = \boldsymbol{T}\boldsymbol{x}$.

The question is how coefficients of a line, resp. a plane are changed without need to recompute them from the definition.

It can be proved that

$$\boldsymbol{p}' = (\boldsymbol{T}\boldsymbol{x}_1) \times (\boldsymbol{T}\boldsymbol{x}_2) = det(\boldsymbol{T})(\boldsymbol{T}^{-1})^T \boldsymbol{p}$$

or

$$\boldsymbol{\rho}' = (\boldsymbol{T}\boldsymbol{x}_1) \times (\boldsymbol{T}\boldsymbol{x}_2) \times (\boldsymbol{T}\boldsymbol{x}_3) = det(\boldsymbol{T})(\boldsymbol{T}^{-1})^T \boldsymbol{\rho}$$

Transformation of lines and planes

As the computation is made **in the projective space** we can write

$$\boldsymbol{p}' = (\boldsymbol{T}^{-1})^T \boldsymbol{p} = [a',\ b' : c']^T \qquad \text{for lines in } E^2$$

or

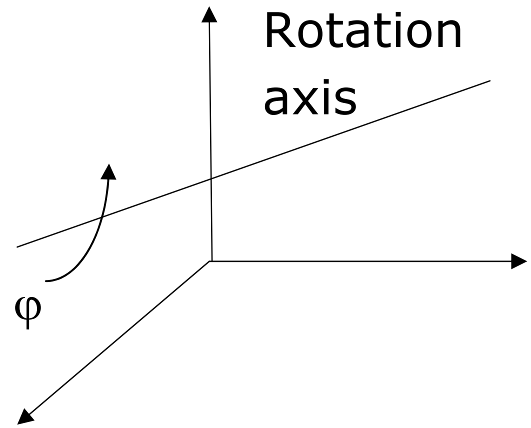
$$\boldsymbol{\rho}' = (\boldsymbol{T}^{-1})^T \boldsymbol{\rho} = [a',\ b', c' : d']^T \qquad \text{for planes in } E^3$$

THIS IS SIMPLIFICATION OF COMPUTATIONS

Transformation matrices for lines, resp. for planes are **DIFFERENT** from transformations for points! Note that a normal vector is actually a co-vector, i.e. an oriented "surface"

Transformation of lines and planes

Transformation about a general axis in E^3 / P^3



Usually used transformation ($\boldsymbol{T}$ is translation):

$$\boldsymbol{Q} = \boldsymbol{T}^{-1}\boldsymbol{R}_{zx}^{-1}\boldsymbol{R}_{yz}^{-1}\boldsymbol{R}(\varphi)\boldsymbol{R}_{zx}\boldsymbol{R}_{xy}\boldsymbol{T}$$

$$\boldsymbol{R}_{yz} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{c}{\sqrt{b^2+c^2}} & \frac{-b}{\sqrt{b^2+c^2}} & 0 \\ 0 & \frac{b}{\sqrt{b^2+c^2}} & \frac{c}{\sqrt{b^2+c^2}} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$\boldsymbol{a} = [a, b, c]^T$ is an axis directional vector. This is unstable if $\sqrt{b^2+c^2} \rightarrow 0$ and not precise if $b^2 \gg c^2$ or vice versa.

That is generally computationally complex and unstable as *a user has to select which axis is to be used for a rotation*

Transformation of lines and planes

Transformation about an axis $\boldsymbol{n}$
in the Euclidean space E^3

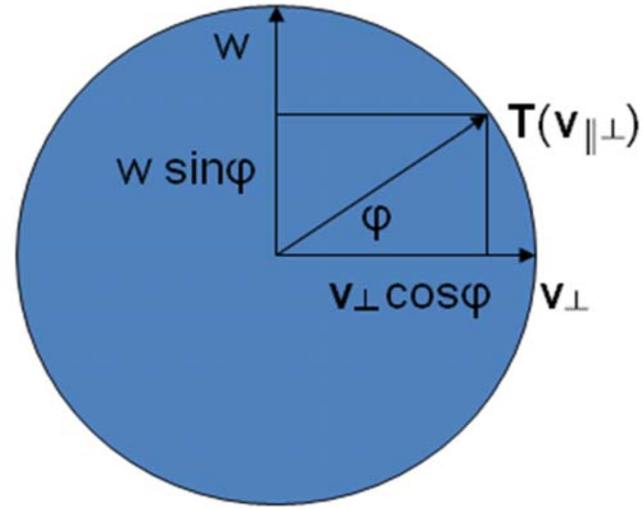
$$\boldsymbol{X} = \boldsymbol{X}\,cos\varphi + (1 - cos\varphi)(\,\boldsymbol{n}^T\boldsymbol{X}).\boldsymbol{n} + (\boldsymbol{n} \times \boldsymbol{X})sin\varphi$$

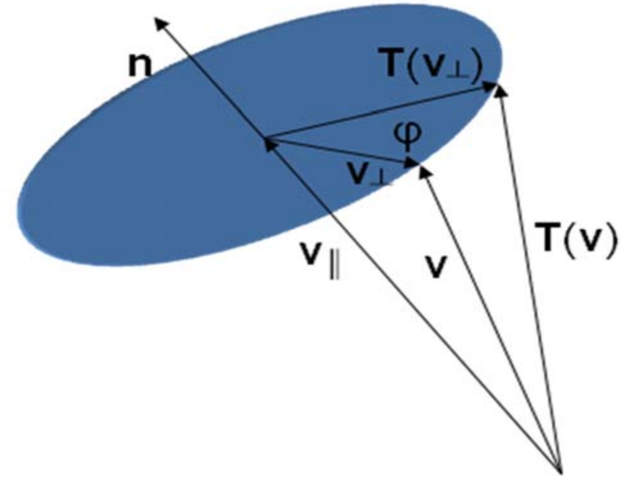
$$\boldsymbol{Q} = \boldsymbol{I}cos\varphi + (1 - cos\varphi)(\boldsymbol{n} \otimes \boldsymbol{n}) + \boldsymbol{W}sin\varphi$$

where: $\boldsymbol{n} \otimes \boldsymbol{n} = \boldsymbol{n}\,.\,\boldsymbol{n}^T$ is a matrix.
In the Euclidean space E^3 the vector $\boldsymbol{n}$ has to be normalized

The matrix $\boldsymbol{W}$ is defined as: $\boldsymbol{W}\boldsymbol{v} = \boldsymbol{w} \times \boldsymbol{v}$

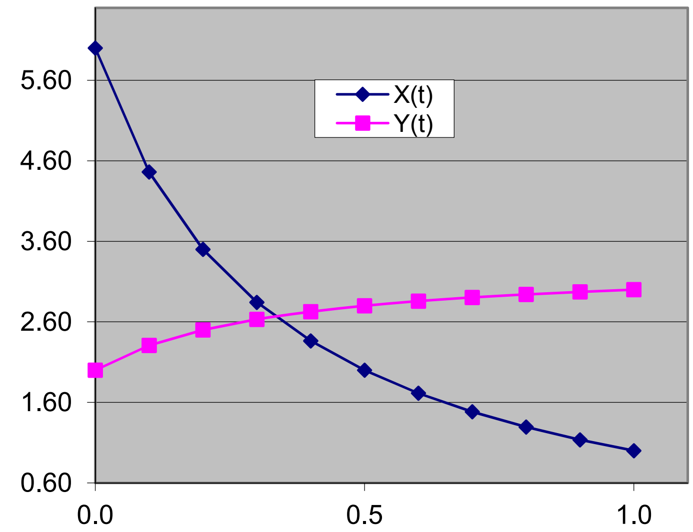
$$\boldsymbol{W} = \begin{bmatrix} 0 & -n_z & n_y \\ n_z & 0 & -n_x \\ -n_y & n_x & 0 \end{bmatrix}$$





Computation in Projective Space

Interpolation



Linear parametrization

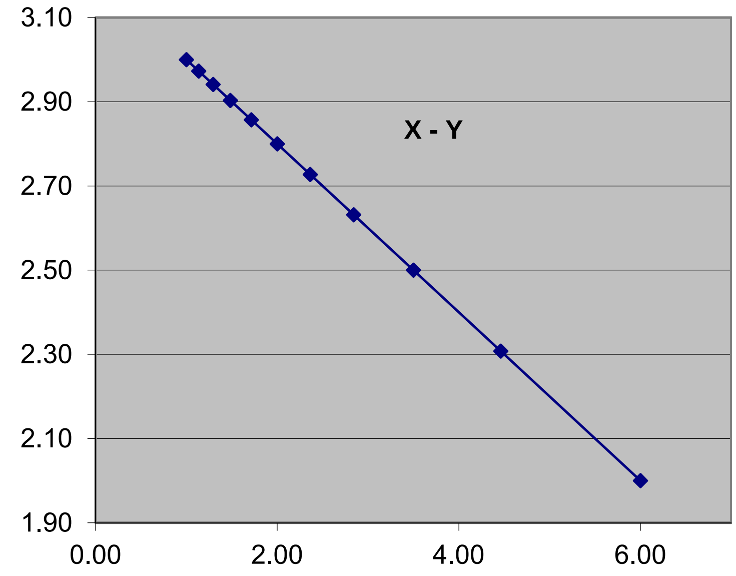
$$\mathbf{X}(t) = \mathbf{X}_0 + (\mathbf{X}_1 - \mathbf{X}_0)t \quad t \in (-\infty, \infty)$$

Non-linear (monotonous) parametrization

$$\mathbf{x}(t) = \mathbf{x}_0 + (\mathbf{x}_1 - \mathbf{x}_0)t \quad t \in (-\infty, \infty)$$

$$x(t) = x_0 + (x_1 - x_0)t \qquad y(t) = y_0 + (y_1 - y_0)t$$

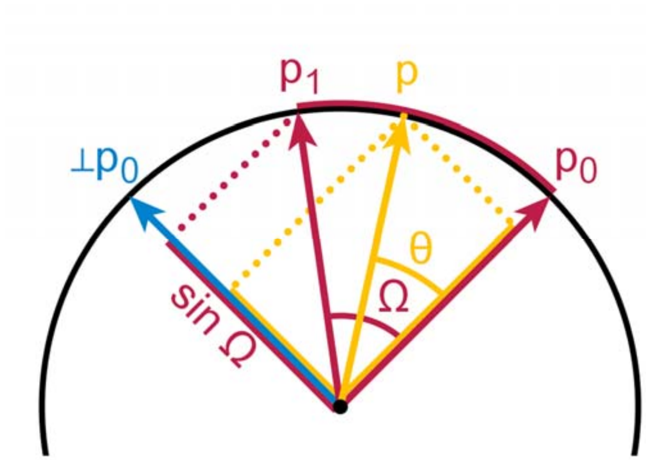
$$z(t) = z_0 + (z_1 - z_0)t \qquad w(t) = w_0 + (w_1 - w_0)t$$



- *t* means that we can interpolate using homogeneous coordinates without a need of "normalization" to E^k !!
- Homogeneous coordinate $w \geq 0$

In many algorithms, we need "monotonous" parameterization, only

Computation in Projective Space



Spherical interpolation

$$slerp(\boldsymbol{X}_0, \boldsymbol{X}_1, t) = \frac{\sin[(1-t)\Omega]}{\sin\Omega}\boldsymbol{X}_0 + \frac{\sin[t\Omega]}{\sin\Omega}\boldsymbol{X}_1$$

Instability occurs if $\Omega \rightarrow k\pi$.

Mathematically formula is correct,
in practice the **code is generally incorrect**! [$\frac{0}{0}$]

$$slerp_p(\boldsymbol{X}_0, \boldsymbol{X}_1, t) = \left[\frac{\sin[(1-t)\Omega]\boldsymbol{X}_0 + \sin[t\Omega]\boldsymbol{X}_1}{\sin\Omega}\right]$$
$$\equiv [\sin[(1-t)\Omega]\boldsymbol{X}_0 + \sin[t\Omega]\boldsymbol{X}_1 : \sin\Omega]^T$$

Homogeneous coordinates
=> better numerical stability

Homogeneous coordinate

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Computation in Projective Space

$$\boldsymbol{x}(t) = slerp(\boldsymbol{x}_0, \boldsymbol{x}_1, t) = \frac{\sin[(1-t)\Omega]}{\sin\Omega}\boldsymbol{x}_0 + \frac{\sin[t\Omega]}{\sin\Omega}\boldsymbol{x}_1$$

What is a result in the Euclidean space of a spherical interpolation with non-linear parameterization?

$x(t) = \frac{\sin[(1-t)\Omega]}{\sin\Omega}x_0 + \frac{\sin[t\Omega]}{\sin\Omega}x_1$	$w(t) = \frac{\sin[(1-t)\Omega]}{\sin\Omega}w_0 + \frac{\sin[t\Omega]}{\sin\Omega}w_1$

$$\boldsymbol{X}(t) = \frac{slerp(x_0, x_1, t)}{slerp(w_0, w_1, t)} = \frac{\frac{\sin[(1-t)\Omega]}{\sin\Omega}x_0 + \frac{\sin[t\Omega]}{\sin\Omega}x_1}{\frac{\sin[(1-t)\Omega]}{\sin\Omega}w_0 + \frac{\sin[t\Omega]}{\sin\Omega}w_1}$$

$$= \frac{\sin[(1-t)\Omega]\,x_0 + \sin[t\Omega]\,x_1}{\sin[(1-t)\Omega]\,w_0 + \sin[t\Omega]\,w_1}$$

If represented as “projective scalar” value

$$\boldsymbol{x}(t) = [\sin[(1-t)\Omega]\,x_0 + \sin[t\Omega]\,x_1 : \sin[(1-t)\Omega]\,w_0 + \sin[t\Omega]\,w_1]^T$$

=> better numerical stability **!!!**

Computation in Projective Space

Intersection line – plane

$$\mathbf{x}(t) = \mathbf{x}_0 + (\mathbf{x}_1 - \mathbf{x}_0)t \qquad t \in (-\infty, \infty)$$

$$\mathbf{a}^T \mathbf{x} = 0 \qquad ax + by + cz + d = 0$$

$$\mathbf{a} = [a, b, c, d]^T \qquad \mathbf{S} = \mathbf{X}_1 - \mathbf{X}_0$$

$$t = -\frac{\mathbf{a}^T \mathbf{x}_0}{\mathbf{a}^T \mathbf{s}}$$

$$\tau = -\mathbf{a}^T \mathbf{x}_0 \qquad \tau_w = \mathbf{a}^T \mathbf{s}$$

$$\mathbf{t} = [\tau : \tau_w] \quad \textbf{if } \tau_w \le 0 \textbf{ then } \mathbf{t} := -\mathbf{t}$$

TEST

$$\textbf{if } t > t_{\min} \textbf{ then}.....$$

$$\textbf{if } \tau * \tau_{\min_w} > \tau_w * \tau_{\min} \textbf{ then}..... \text{ condition } \tau \ge 0$$

An intersection of a plane with a line in E^2 / E^3 can be computed efficiently

Comparison operations must be modified!!!

Cyrus-Beck line clipping algorithm 10-25% faster

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Line Clipping (for a convex polygon)

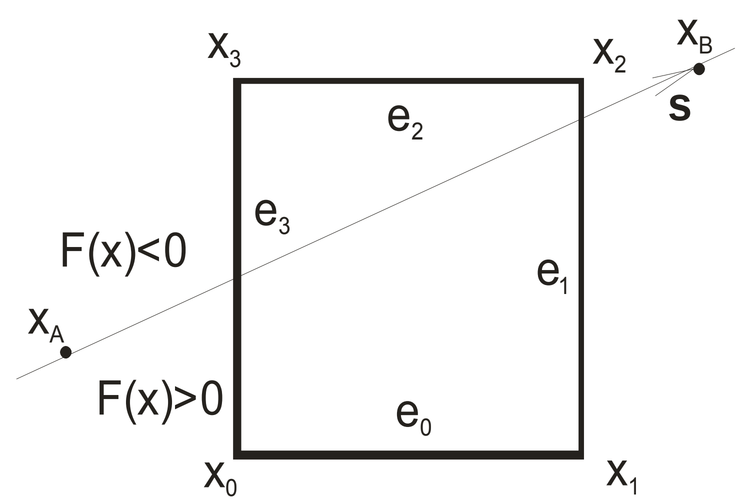
procedure CLIP_Line ($\boldsymbol{x}_A$, $\boldsymbol{x}_B$);
{ **input**: $\boldsymbol{x}_A = [x_A, y_A : 1]^T$ $\boldsymbol{p} = [a, b : c]^T$}
begin { N = 4 }
{1}$\boldsymbol{p} := \boldsymbol{x}_A \times \boldsymbol{x}_B$; { p: ax+by+c = 0}
{2}**for** k:=0 **to** N-1 **do** { $\boldsymbol{x}_k = [x_k, y_k, 1]^T$ }
{3} **if** $\boldsymbol{p}^T \boldsymbol{x}_k \geq 0$ **then** $\mathbf{c}_k := 1$ **else** $c_k := 0$;
{4}**if** **c** = $[0000]^T$ **or c** = $[1111]^T$ **then EXIT**;
{5}i:= TAB1[**c**]; j:= TAB2[**c**];
{6}$\boldsymbol{x}_A := \boldsymbol{p} \times \boldsymbol{e}_i$; $\boldsymbol{x}_B := \boldsymbol{p} \times \boldsymbol{e}_j$;
{7}**DRAW** ($\boldsymbol{x}_A$; $\boldsymbol{x}_B$) {$\boldsymbol{e}_i$ – *i-th edge* }
end {CLIP_Line}

SIMPLE, ROBUST and FAST

- **Simplification for a window <-1,-1>x<1, 1>**

Line clipping algorithms in E^2

- Cohen-Sutherland
- Liang-Barsky
- Hodgman
- Skala – modification of Clip_L for line segments



- Skala,V.: A new approach to line and line segment clipping in homogeneous coordinates, The Visual Computer, SpringerVol.21, No.11, pp.905-914, 2005

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/* Additional code for Line Segment clipping colored */

function CODE (**x**); /* Cohen Sutherland window coding */

begin **c**:= [0000]; /* **land** / **lor** – bitwise operations **and** / **or** */

    if $x<x_{min}$ **then** **c**:=[1000] **else if** $x>x_{max}$ **then** **c**:= [0100];

    if $y<y_{min}$ **then** **c**:=**c** **lor** [1001] **else if** $y>y_{max}$ **then** **c**:=**c** **lor** [0010];

    CODE := **c**

end /*CODE */;

procedure Clip_LineSegment ($\mathbf{x}_A$, $\mathbf{x}_B$); /* short x long line segments */

begin $\mathbf{c}_A$:= CODE ($\mathbf{x}_A$); $\mathbf{c}_B$:= CODE ($\mathbf{x}_B$);

    if ($\mathbf{c}_A$ **lor** $\mathbf{c}_B$) = **0** **then** { **output** ($\mathbf{x}_A$; $\mathbf{x}_B$); **EXIT** } /* inside */

    if ($\mathbf{c}_A$ **land** $\mathbf{c}_B$) $\neq$ **0** **then EXIT;** /* outside */

    $\mathbf{p} := \mathbf{x}_A \times \mathbf{x}_B$; /* $ax+by+c = 0$; $\mathbf{p} = [a,b:c]^T$ */

    for k:=0 **to** 3 **do** /* $\mathbf{x}_k=[x_k,y_k:1]^T$ $\mathbf{c}=[\ c_3, c_2, c_1, c_0\ ]^T$ */

        if $\mathbf{p}^T\mathbf{x}_k \geq 0$ **then** c_k:=1 **else** c_k:=0;

    if **c** = $[0000]^T$ **or** **c** = $[1111]^T$ **then EXIT**;

    i:= TAB1[**c**]; j:= TAB2[**c**];

    /* original code $\mathbf{x}_A := \mathbf{p} \times \mathbf{e}_i$; $\mathbf{x}_B := \mathbf{p} \times \mathbf{e}_j$; **DRAW** ($\mathbf{x}_A$; $\mathbf{x}_B$) */

/* additional code for Line Segment clipping */

if $c_A \neq 0$ **and** $c_B \neq 0$

then { $x_A := p \times e_i$; $x_B := p \times e_j$ } /* two intersections */

else { there is only one intersection point }

if $c_A = 0$ **then** { x_B is outside }

{ **if** c_B **land** MASK[**c**] $\neq 0$

then $x_B := p \times e_i$

else $x_B := p \times e_j$

}

else if $c_B = 0$ **then**

{ x_A is outside }

{ **if** c_A **land** MASK[**c**] $\neq 0$

then $x_A := p \times e_i$

else $x_A := p \times e_j$

};

output (x_A , x_B)

end {Clip_LineSegment}

Algorithm can be extended to convex polygon clipping & modified for parametric lines as well

c	c_3	c_2	c_1	c_0	**TAB1**	**TAB2**	**MASK**
0	0	0	0	0	None	None	None
1	0	0	0	1	0	3	0100
2	0	0	1	0	0	1	0100
3	0	0	1	1	1	3	0010
4	0	1	0	0	1	2	0010
5	0	1	0	1	N/A	N/A	N / A
6	0	1	1	0	0	2	0100
7	0	1	1	1	2	3	1000
8	1	0	0	0	2	3	1000
9	1	0	0	1	0	2	0100
10	1	0	1	0	N/A	N/A	N / A
11	1	0	1	1	1	2	0010
12	1	1	0	0	1	3	0010
13	1	1	0	1	0	1	0100
14	1	1	1	0	0	3	0100
15	1	1	1	1	None	None	None

- Skala,V.: S-Clip E2: A New Concept of Clipping Algorithms, Poster, SIGGRAPH Asia 2012

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Computation in Projective Space - Barycentric coordinates

Let us consider a triangle with vertices $\mathbf{X}_1$, $\mathbf{X}_2$, $\mathbf{X}_3$,

A position of any point $\mathbf{X} \in E^2$ can be expressed as

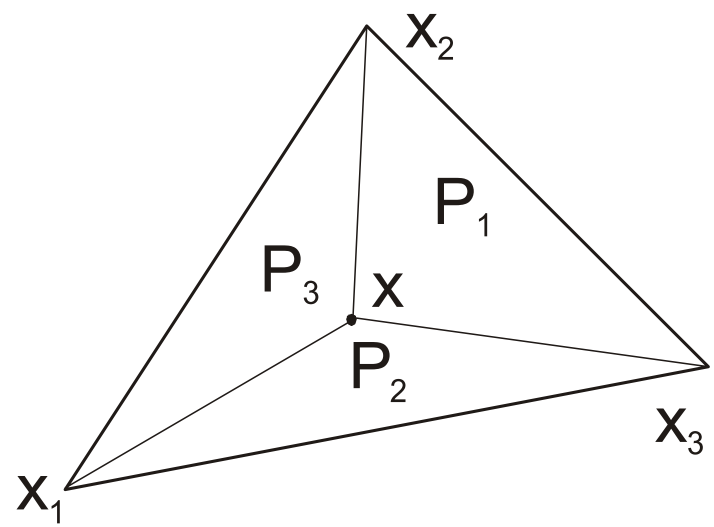
$$a_1 X_1 + a_2 X_2 + a_3 X_3 = X$$

$$a_1 Y_1 + a_2 Y_2 + a_3 Y_3 = Y$$

additional condition

$$a_1 + a_2 + a_3 = 1 \quad 0 \le a_i \le 1$$

$$a_i = \frac{P_i}{P} \quad \mathrm{i} = 1,\ldots,3$$



A linear system of equations $\boldsymbol{Ax} = \boldsymbol{b}$ has to be solved

If points $\mathbf{x}_i$ are given as $[x_i, y_i, z_i: w_i]^T$ and $w_i \neq 1$ then $\mathbf{x}_i$ must be "normalized" to $w_i = 1$, i.e. 4 * 3 = 12 division operations

Computation in Projective Space

$$b_1 X_1 + b_2 X_2 + b_3 X_3 + b_4 X = 0$$

$$b_1 Y_1 + b_2 Y_2 + b_3 Y_3 + b_4 Y = 0$$

$$b_1 + b_2 + b_3 + b_4 = 0$$

$$b_i = -a_i b_4 \quad i = 1, \dots, 3 \quad b_4 \neq 0$$

Rewriting

$$\begin{bmatrix} X_1 & X_2 & X_3 & X \\ Y_1 & Y_2 & Y_3 & Y \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} = \mathbf{0}$$

$$\mathbf{b} = \boldsymbol{\xi} \times \boldsymbol{\eta} \times \mathbf{w}$$

$$\mathbf{b} = [b_1, b_2, b_3, b_4]^T$$

$$\boldsymbol{\xi} = [X_1, X_2, X_3, X]^T$$

$$\boldsymbol{\eta} = [Y_1, Y_2, Y_3, Y]^T$$

$$\mathbf{w} = [1,1,1,1]^T$$

Solution of the linear system of equations (LSE) is equivalent to generalized cross product

$$\boldsymbol{b} = \boldsymbol{\xi} \times \boldsymbol{\eta} \times \boldsymbol{w}$$

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Computation in Projective Space

if $w_i \neq 1$ or $w_i = 1$

$$\begin{bmatrix} x_1 & x_2 & x_3 & x \\ y_1 & y_2 & y_3 & y \\ w_1 & w_2 & w_3 & w \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} = \mathbf{0}$$

$$\mathbf{b} = \boldsymbol{\xi} \times \boldsymbol{\eta} \times \mathbf{w}$$

$$\mathbf{b} = \left[b_1, b_2, b_3, b_4 \right]^T$$

$$\boldsymbol{\xi} = \left[x_1, x_2, x_3, x \right]^T$$

$$\boldsymbol{\eta} = \left[y_1, y_2, y_3, y \right]^T$$

$$\mathbf{w} = \left[w_1, w_2, w_3, w \right]^T$$

$$0 \leq (-b_1 : w_2 w_3 w) \leq 1$$

$$0 \leq (-b_2 : w_3 w_1 w) \leq 1$$

$$0 \leq (-b_3 : w_1 w_2 w) \leq 1$$

=> **new entities**:

projective scalar, projective vector

- Skala,V.: Barycentric coordinates computation in homogeneous coordinates, Computers&Graphics, 2008)

Computation in Projective Space

Line in E3 as Two Plane Intersection

Standard formula in the Euclidean space

$$\boldsymbol{\rho}_1 = [a_1, b_1, c_1 : d_1]^T = [\boldsymbol{n}_1^T : d_1]^T \qquad \boldsymbol{\rho}_2 = [a_2, b_2, c_2 : d_2]^T = [\boldsymbol{n}_2^T : d_2]^T$$

Line given as an intersection of two planes

$$\boldsymbol{s} = \boldsymbol{n}_1 \times \boldsymbol{n}_2 \equiv [a_3, b_3, c_3 : 0]^T \qquad \boldsymbol{X}(\boldsymbol{t}) = \boldsymbol{X}_0 + \boldsymbol{s}t$$

$$X_0 = \frac{d_2 \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} - d_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}}{DET} \qquad Y_0 = \frac{d_2 \begin{vmatrix} a_3 & c_3 \\ a_1 & c_1 \end{vmatrix} - d_1 \begin{vmatrix} a_3 & c_3 \\ a_2 & c_2 \end{vmatrix}}{DET}$$

$$Z_0 = \frac{d_2 \begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix} - d_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}}{DET} \qquad DET = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

The formula is quite “horrible” one and for students not acceptable as it is too complex and they do not see from the formula comes from.

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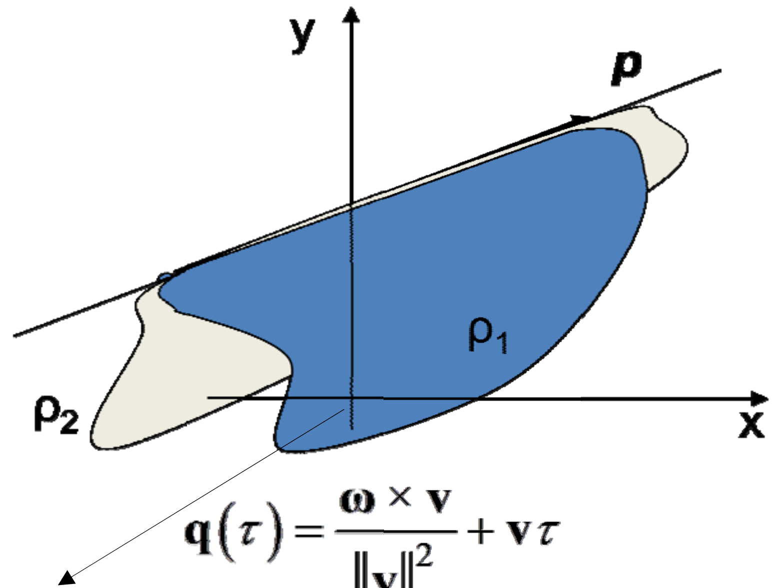
Computation in Projective Space

Line in E3 as Two Plane Intersection

- Standard formula in the Euclidean space
- Plücker’s coordinates – a line is given by two points. Due to the DUALITY - point is dual to a plane and vice versa – the intersection of two planes can be computed as a dual problem – but computationally **expensive** computation



$$\mathbf{q}(\tau) = \frac{\boldsymbol{\omega} \times \mathbf{v}}{\|\mathbf{v}\|^2} + \mathbf{v}\tau$$

$$\boldsymbol{\omega} = \begin{bmatrix} l_{41} & l_{42} & l_{43} \end{bmatrix}^T \quad \mathbf{v} = \begin{bmatrix} l_{23} & l_{31} & l_{12} \end{bmatrix}^T$$

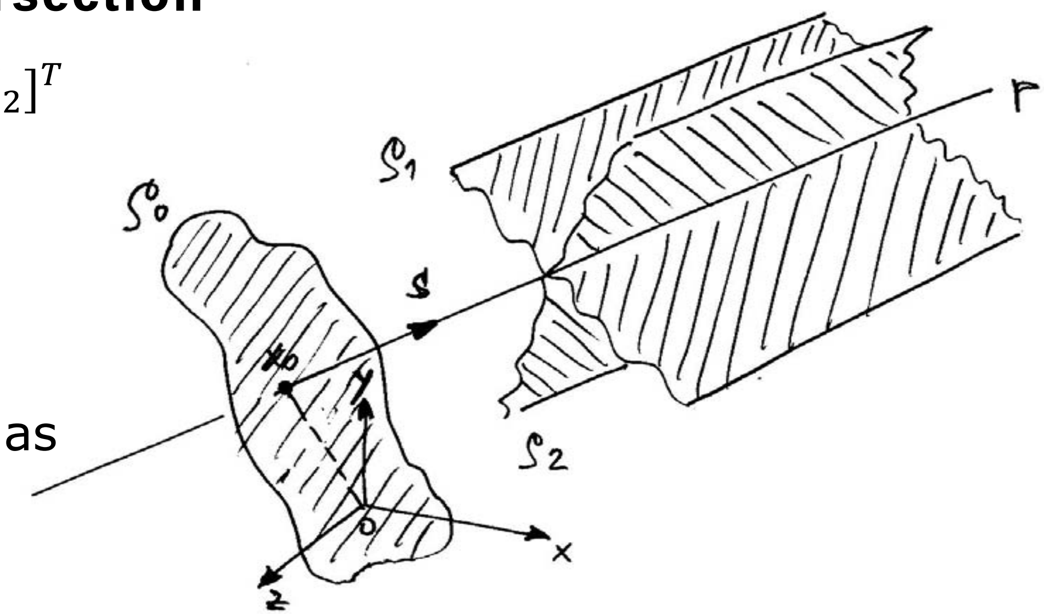
$$\mathbf{L} = \mathbf{a}_0\mathbf{a}_1^T - \mathbf{a}_1\mathbf{a}_0^T \quad \textit{tensor product} - \textit{matrix}$$

$$\mathbf{a}_i = \left[a_i, b_i, c_i, d_i\right]^T \quad i = 1,2$$

- Projective formulation and simple computation

Computation in Projective Space

Line in E3 as Two Plane Intersection



$\boldsymbol{\rho}_1 = [a_1, b_1, c_1 : d_1]^T$ $\boldsymbol{\rho}_2 = [a_2, b_2, c_2 : d_2]^T$

normal vectors are

$\boldsymbol{n}_1 = [a_1, b_1, c_1]^T$ $\boldsymbol{n}_2 = [a_2, b_2, c_2]^T$

Directional vector of a line
of two planes $\boldsymbol{\rho}_1$ and $\boldsymbol{\rho}_1$ is given as

$$\boldsymbol{s} = \boldsymbol{n}_1 \times \boldsymbol{n}_2$$

"starting" point $\boldsymbol{x}_0$ **???**

A plane $\boldsymbol{\rho}_0$ passing the origin with a normal vector $\boldsymbol{s}$, $\boldsymbol{\rho}_0 = [a_0, b_0, c_0 : 0]^T$

The point $\boldsymbol{x}_0$ is defined as $\boldsymbol{x}_0 = \boldsymbol{\rho}_1 \times \boldsymbol{\rho}_2 \times \boldsymbol{\rho}_0$

Simple formula for matrix-vector architectures like GPU and parallel processing. *Compare the standard and projective formulas*

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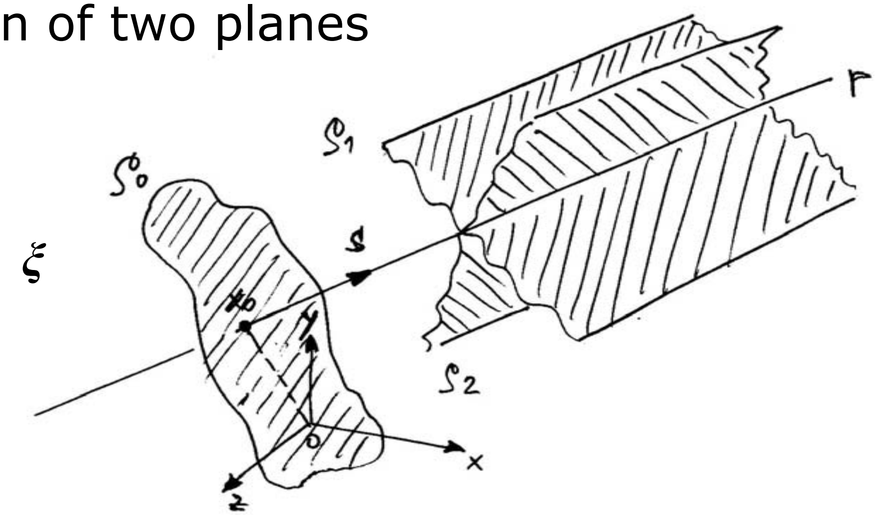
Computation in Projective Space – the nearest point

Find the nearest point on an intersection of two planes to the given point ξ



Simple solution:

- Translate planes $\boldsymbol{\rho}_1$ and $\boldsymbol{\rho}_2$ so the ξ is in the origin
- Compute intersection of two planes i.s. $\boldsymbol{x}_0$ and $\boldsymbol{s}$
- Translate $\boldsymbol{x}_0$ using $\boldsymbol{T}^{-1}$

Again – an elegant solution, simple formula supporting matrix-vector architectures like GPU and parallel processing

Solution DETAILS next

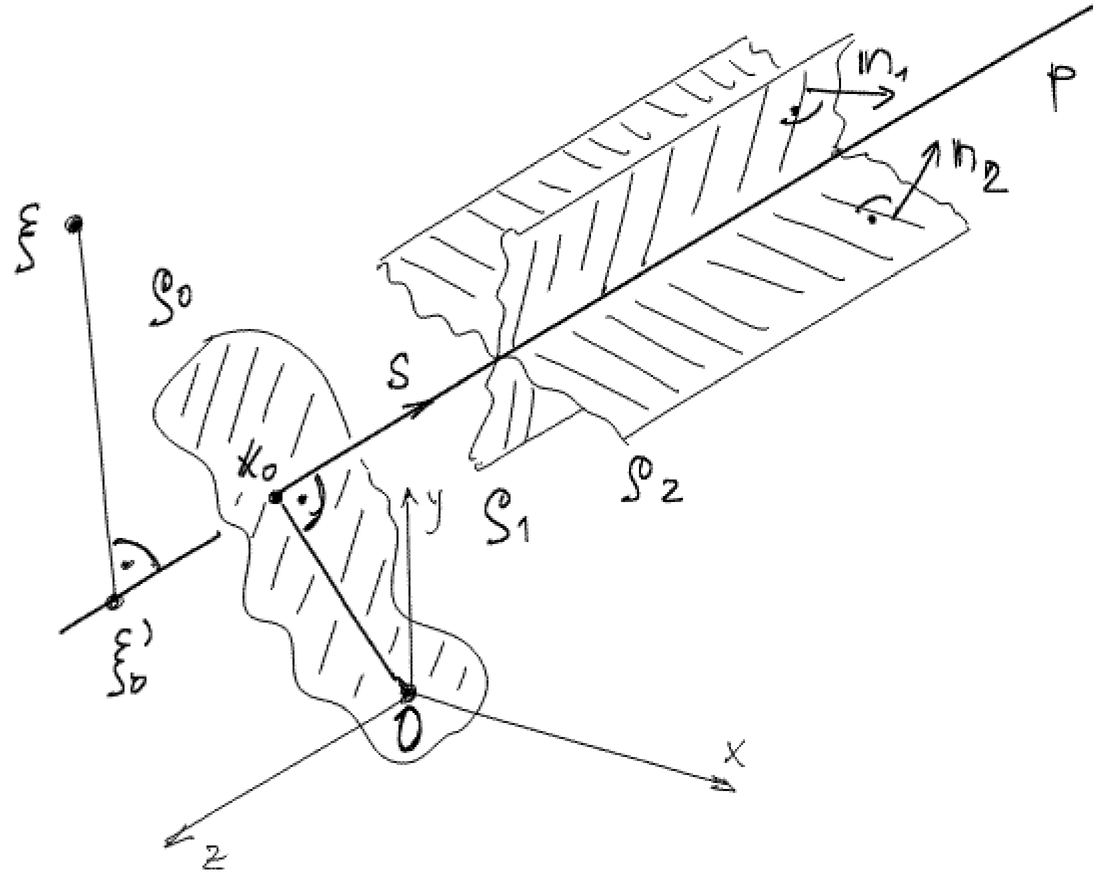
The known solution using Lagrange multipliers

$$\begin{bmatrix} 2 & 0 & 0 & n_{1x} & n_{2x} \\ 0 & 2 & 0 & n_{1y} & n_{2y} \\ 0 & 0 & 2 & n_{1z} & n_{2z} \\ n_{1x} & n_{1y} & n_{1z} & 0 & 0 \\ n_{2x} & n_{2y} & n_{2z} & 0 & 0 \end{bmatrix} \begin{pmatrix} p_x \\ p_y \\ p_z \\ \lambda \\ \mu \end{pmatrix} = \begin{pmatrix} 2p_{0x} \\ 2p_{0y} \\ 2p_{0z} \\ \vec{p}_1 \cdot \vec{n}_1 \\ \vec{p}_2 \cdot \vec{n}_2 \end{pmatrix}$$

Krumm,J.: Intersection of Two Planes, Microsoft Research

The closest point to an intersection of two planes

In some applications we need to find a closest point on a line given as an intersection of two planes. We want to find a point $\boldsymbol{\xi}_0{}'$, the closest point to the given point $\boldsymbol{\xi}$, which lies on an intersection of two planes



$\boldsymbol{\rho}_1 \triangleq [\boldsymbol{n}_1^T : d_1]^T$ and $\boldsymbol{\rho}_2 \triangleq [\boldsymbol{n}_2^T : d_2]^T$

This problem was recently solved by using Lagrange multipliers and an optimization approach leading to a solution of a system of linear equations with 5 equations.

Solution in the projective space

1. Translate the given point $\boldsymbol{\xi} = [\xi_x, \xi_y, \xi_z : 1]^T$ to the origin – matrix $\boldsymbol{Q}$
2. Compute parameters of the given planes $\boldsymbol{\rho}_1$ and $\boldsymbol{\rho}_2$ after the transformation as $\boldsymbol{\rho}_1' = \boldsymbol{Q}^{-T}\boldsymbol{\rho}_1$ and $\boldsymbol{\rho}_2' = \boldsymbol{Q}^{-T}\boldsymbol{\rho}_2$,
3. Compute the intersection of those two planes $\boldsymbol{\rho}_1'$ and $\boldsymbol{\rho}_2'$
4. Transform the point $\boldsymbol{\xi}_0$ to the original coordinate system using transformation

$$\boldsymbol{n}_0 = \boldsymbol{n}_1 \times \boldsymbol{n}_2 \qquad \boldsymbol{\rho}_0 \triangleq [\boldsymbol{n}_0^T : 0]^T \qquad \boldsymbol{\xi}_0 = \boldsymbol{\rho}_1 \times \boldsymbol{\rho}_2 \times \boldsymbol{\rho}_0 \qquad \boldsymbol{\xi}_0' = \boldsymbol{Q}^{-1}\,\boldsymbol{\xi}_0$$

$$\boldsymbol{Q} = \begin{bmatrix} 1 & 0 & 0 & -\xi_x \\ 0 & 1 & 0 & -\xi_y \\ 0 & 0 & 1 & -\xi_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \boldsymbol{Q}^{-T} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \xi_x & \xi_y & \xi_z & 1 \end{bmatrix} \qquad \boldsymbol{Q}^{-1} = \begin{bmatrix} 1 & 0 & 0 & \xi_x \\ 0 & 1 & 0 & \xi_y \\ 0 & 0 & 1 & \xi_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

It is simple, easy to implement on GPU.

Computation in Projective Space

Disadvantages

- Careful handling with formula as the projective space
- “Oriented” projective space is to be used, i.e. $w \geq 0$; HW could support it
- Exponents of the homogeneous vector can overflow
  - o exponents should be normalized; HW could support it
  - o unfortunately not supported by the current hardware
  - o P_Lib – library for computation in the projective space - uses SW solution for normalization on GPU (C# and C++)

Computation in Projective Space

Advantages

- "Infinity" is represented
- No division operation is needed, a division operation can be hidden to the homogeneous coordinate
- Many mathematical formula are simpler and elegant
- One code sequence solve primary and dual problems
- Supports matrix – vector operations in hardware – like GPU etc.
- Numerical computation can be faster
- More robust and stable solutions can be achieved
- System of linear equations can be solved directly without division operation if exponent normalization is provided
- If the cross product, i.e. $\boldsymbol{x}_1 \times \ldots \times \boldsymbol{x}_n$, is used instead of $\boldsymbol{Ax} = \boldsymbol{b}$, the formula $\boldsymbol{x}_1 \times \ldots \times \boldsymbol{x}_n$ can be used for further symbolic manipulation

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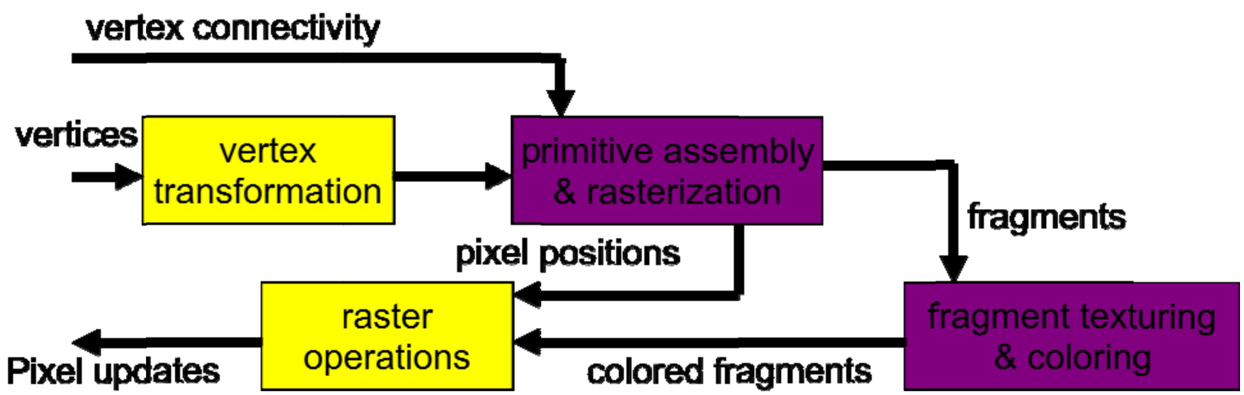
Implementation aspects and GPU

- GPU (Graphical Processing Unit) -optimized for matrix-vector, vector-vector operation – especially for [x,y,z :w]
- Native arithmetic operations with homogeneous coordinates – without exponent “normalization”
- Programmable HW – parallel processing



Implementation aspects and GPU

4D cross product can be implemented in Cg/HLSL on GPU (not optimal implementation) as:

```
float4 cross_4D(float4 x1, float4 x2, float4 x3)
{ float4 a; # simple formula #
  a.x=dot(x1.yzw, cross(x2.yzw, x3.yzw));
  a.y=-dot(x1.xzw, cross(x2.xzw, x3.xzw));
  a.z=dot(x1.xyw, cross(x2.xyw, x3.xyw));
  a.w=-dot(x1.xyz, cross(x2.xyz, x3.xyz));
  return a;
}
# more compact formula is available #
```

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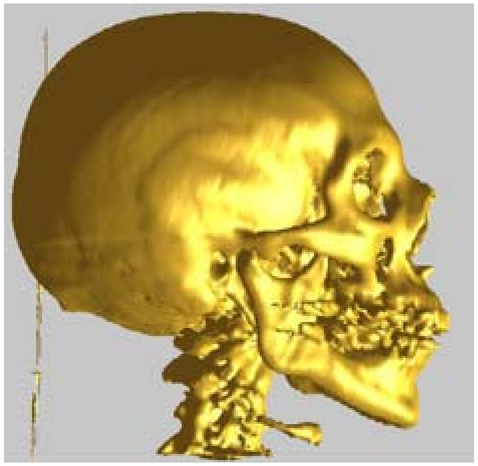

Data processing - main field in computer science

Data processing itself can be split to two main areas:

- **processing of textual data**
  limited interval of values, unlimited dimensionality
  [char as one dimension - Methionylthreonylthreonylglutaminylarginyl...isoleucine   189,819 chars] - no interpolation is defined
- **processing of numerical data**
  unlimited interval of values, limited dimensionality – usually 2 or 3

- interpolation **can be used**

	Textual	Graphical
Dim	∞	2, 3
Interval	0-255 (ASCII)	(-∞, ∞)

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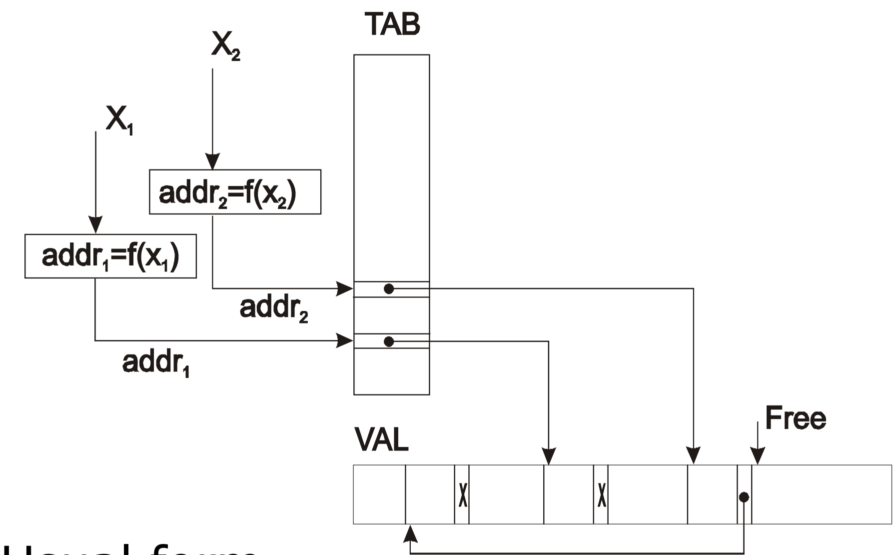
Hash functions

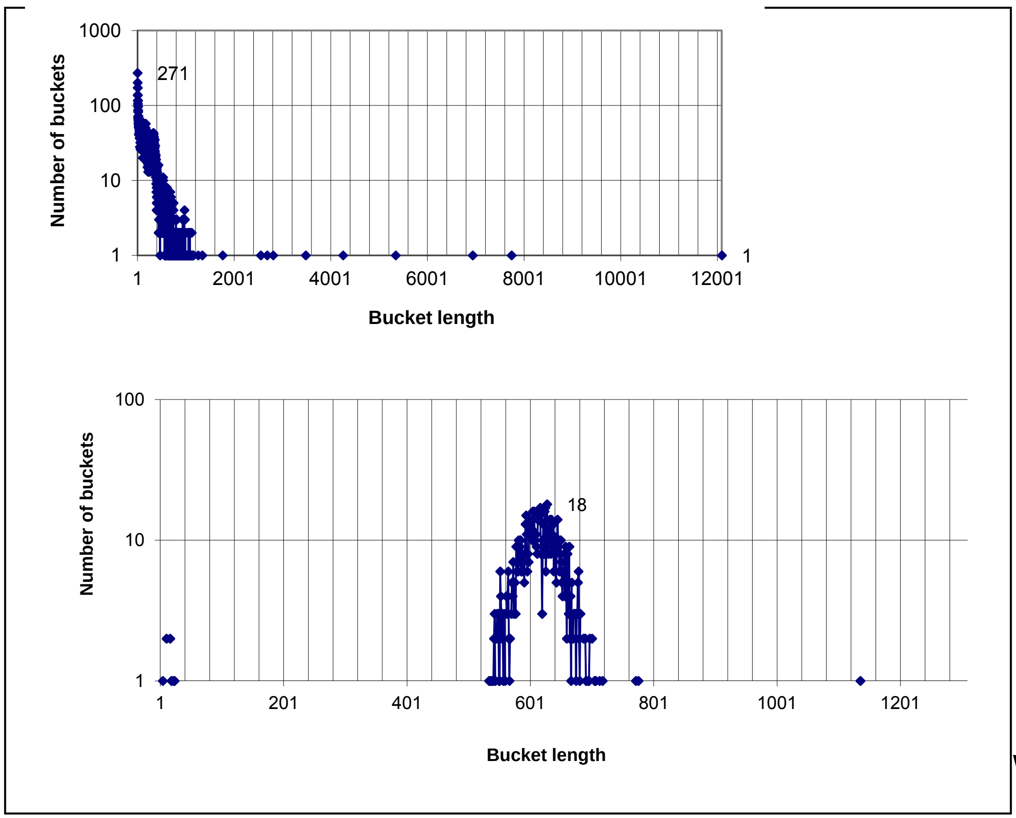
- usually used for textual data processing
- prime numbers and modulo operations are used

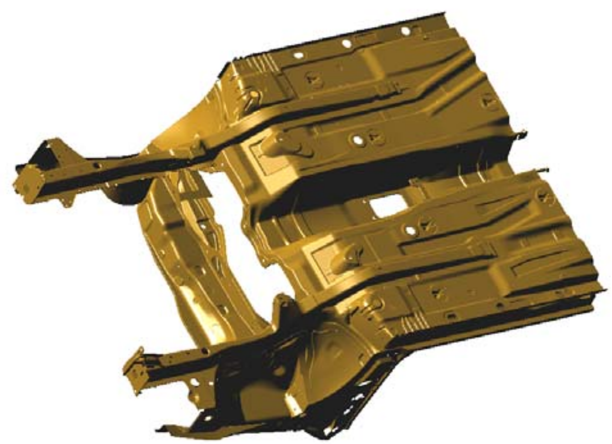


Usual form

$$Addr = \lfloor 3x + 5y + 7z \rfloor \mathbf{mod}\ size$$

multiplication int * float needed





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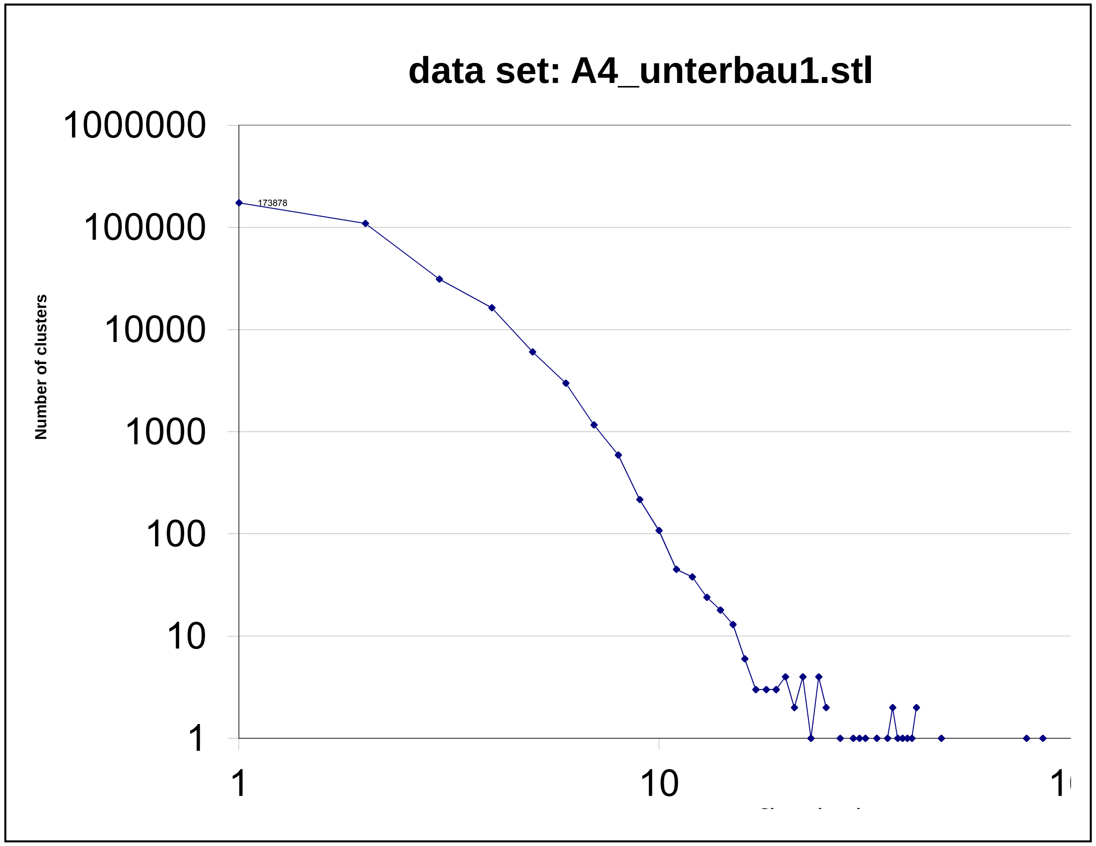
If the hash function is constructed as

$$Addr = \lfloor \alpha x + \beta y + \gamma z \rfloor \mathbf{mod}\ m$$

where α, β, γ are "irrational" numbers and $m = 2^k$ better distribution is obtained => much faster processing.



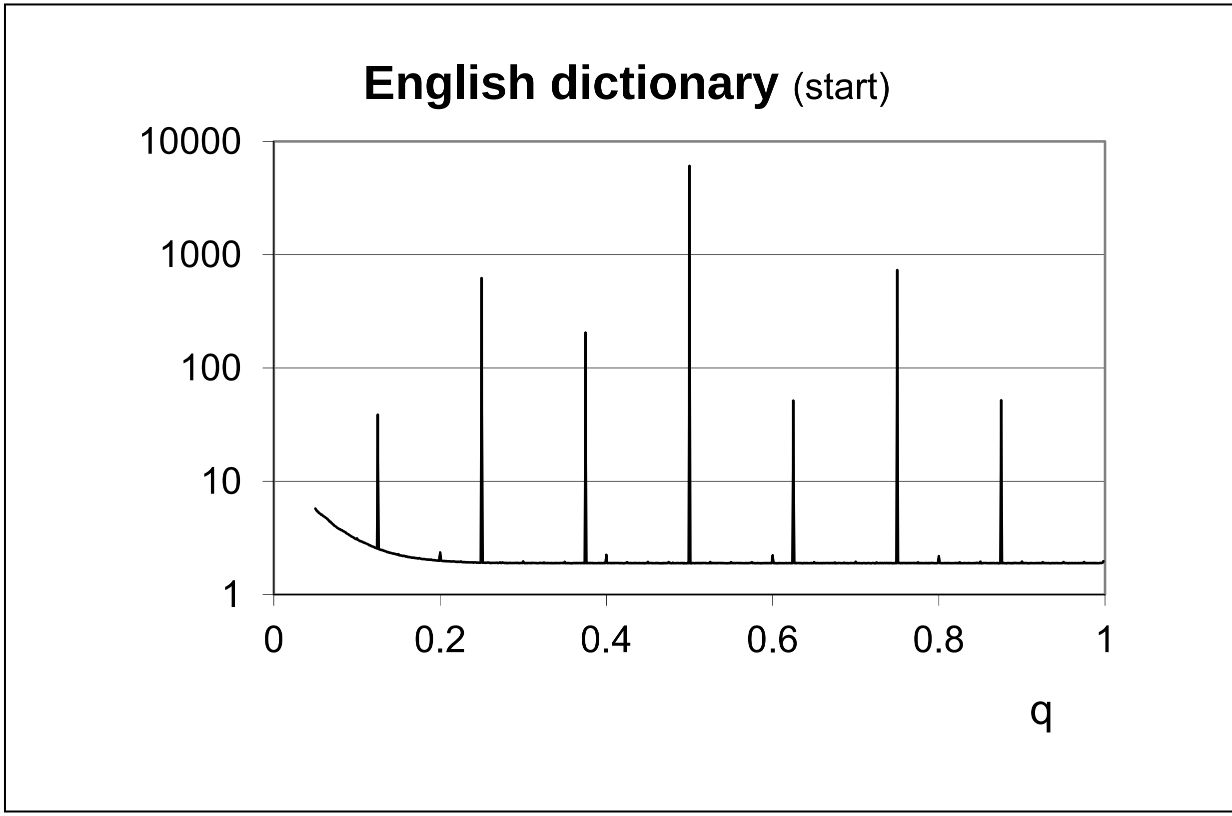
Textual processing

The has function is constructed as

$$h(\boldsymbol{x}) = \left(C * \sum_{i=1}^{L} q^i x_i \right) \boldsymbol{mod}\, m$$

q „irrational“ $0 < q < 1$

$m = 2^k - 1$



Both geometrical and textual hash function design have the same approach coefficients are “irrational” and no division operation is needed.

Some differences for Czech, Hebrew, English, German, Arabic, … languages and “chemical” words.

Geometry algebra

$\boldsymbol{ab} = \boldsymbol{a} \cdot \boldsymbol{b} + \boldsymbol{a} \wedge \boldsymbol{b}$ in E^3 $\boldsymbol{ab} = \boldsymbol{a} \cdot \boldsymbol{b} + \boldsymbol{a} \times \boldsymbol{b}$

It is strange – result of a dot product is a scalar value while result of the outer product (cross product) is a vector.

What is $\boldsymbol{ab}$**???**

Please, for details see

- http://geometricalgebra.zcu.cz/
- GraVisMa – recent workshops on Computer Graphics, Computer Vision & Mathematics http://www.GraVisMa.eu
- **WSCG – Conferences on Computer Graphics, Computer Vision & Visualization** since 1992 http://www.wscg.eu

Summary and conclusion

We have got within this course an understanding of:

- projective representation use for geometric transformations with points, lines and planes
- principle of duality and typical examples of dual problems, influence to computational complexity
- intersection computation of two planes in E3, dual Plücker coordinates and simple projective solution
- geometric problems solution with additional constrains
- intersection computations and interpolation algorithms directly in the projective space
- barycentric coordinates computation on GPU
- avoiding or postponing division operations in computations

Projective space representation supports matrix-vector architectures like GPU – faster, robust and easy to implement algorithms achieved

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Questions

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Supported by the Ministry of Education of the Czech Republic, projects ME10060, LA10035, LH12181