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GEOMETRIC COMPUTATION, DUALITY AND PROJECTIVE SPACE

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This paper presents solutions of some selected problems that can be easily solved in the projective space. If the principle of duality is used, quite surprising solutions can be found and new useful theorems can be generated as well. Principle of duality can be used to derive new solutions, e.g. an equation of a parametric line in E^3 as an intersection of two planes etc. This new formulations avoid division operations and increases the robustness of computation. The homogeneous coordinates are mostly introduced with geometric transformations concept and used for the projective space representation. Nevertheless, geometrical interpretation is missing in many publications. Conversion from the homogeneous coordinates to the Euclidean coordinates is defined for the E^2 case as: $X = x / w \qquad Y = y / w$, where: $w \neq 0$, point $\mathbf{x} = [x, y, w]^T$ and $\mathbf{x} \in P^2$, $\mathbf{X} = [X, Y]^T$ and $\mathbf{X} \in E^2$. If $w = 0$ then $\mathbf{x}$ represents "an ideal point", that is a point in infinity. The presented approach offers higher computational robustness and speed-up as well, especially if GPU technology is used.

The cross-product of two vectors $\mathbf{x}_1$, $\mathbf{x}_2$, if given in the homogeneous coordinates, is defined as:

$$\mathbf{x}_1 \times \mathbf{x}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_1 & y_1 & w_1 \\ x_2 & y_2 & w_2 \end{vmatrix}$$

Let two points $\mathbf{x}_1$, $\mathbf{x}_2$, be given in the projective space. Then a line $\mathbf{p} \in E^2$ defined by those two points is determined as: $\mathbf{p} = \mathbf{x}_1 \times \mathbf{x}_2$

$$\mathbf{p} = [a, b, c]^T \qquad ax + by + c = 0$$

The cross-product of three vectors $\mathbf{x}_1$, $\mathbf{x}_2$ and $\mathbf{x}_3$ is defined in the homogeneous coordinates as:

$$\mathbf{x}_1 \times \mathbf{x}_2 \times \mathbf{x}_3 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} & \mathbf{l} \\ x_1 & y_1 & z_1 & w_1 \\ x_2 & y_2 & z_2 & w_2 \\ x_3 & y_3 & z_3 & w_3 \end{vmatrix}$$

Let three points $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$ be given in the projective space. Then a plane $\boldsymbol{\rho} \in E^3$ defined by those three points is determined as:

$$\boldsymbol{\rho} = \mathbf{x}_1 \times \mathbf{x}_2 \times \mathbf{x}_3 \qquad ax + by + cz + d = 0$$

Now we can use the **principle of duality** as follows:

- As a point is dual to a line, a line is dual to a point in E^2, we can determine an intersection of two lines directly as: $\mathbf{x} = \mathbf{p}_1 \times \mathbf{p}_2$
- As a point is dual to a plane, a plane is dual to a point in E^3, we can determine an intersection of three planes directly as: $\mathbf{x} = \boldsymbol{\rho}_1 \times \boldsymbol{\rho}_2 \times \boldsymbol{\rho}_3$.

It means that we have **ONLY one procedure** which allows us to compute a) a line given by two points, b) an intersection point of two lines, c) a plane given by three points, b) an intersection point of three planes and the algorithm is robust as **the division operation is not used** !

From the above it can be seen that *generalized cross product is "equivalent" to a solution of linear equations* $\mathbf{A}\mathbf{x} = \mathbf{b}$ that could be useful in some computations as well.

In many geometric computations, a line $\mathbf{p} \in E^3$ is given as an intersection of two planes $\boldsymbol{\rho}_1$, $\boldsymbol{\rho}_2$. Fortunately the **Plücker** representation could help us to resolve this in an elegant way directly in the projective space using the principal duality. Traditionally, the line $\mathbf{p}$ is determined by a formula which is "not simple", not easy to remember and to compute. Also as the division operation is used there might be problems with a robustness of the computation.

$\boldsymbol{\rho}_1 : \mathbf{n}_1{}^T\mathbf{x} + d_1 = 0$ $\quad$ $\mathbf{q}(t) = \mathbf{q}_0 + \mathbf{n}_3 t \quad \mathbf{n}_3 = \mathbf{n}_1 \times \mathbf{n}_2$

$\boldsymbol{\rho}_2 : \mathbf{n}_2{}^T\mathbf{x} + d_2 = 0$ $\quad$ $\boldsymbol{q}_0 = [X_0, Y_0, Z_0]^T$

$$DET = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$X_0 = \frac{d_2\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} - d_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}}{DET} \qquad Y_0 = \frac{d_2\begin{vmatrix} a_3 & c_3 \\ a_1 & c_1 \end{vmatrix} - d_1\begin{vmatrix} a_3 & c_3 \\ a_2 & c_2 \end{vmatrix}}{DET} \qquad Z_0 = \frac{d_2\begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix} - d_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}}{DET}$$

Let $\boldsymbol{X}_2 - \boldsymbol{X}_1 = \omega$ and $\boldsymbol{X}_1 \times \boldsymbol{X}_2 = \boldsymbol{v}$. A point on the line $\boldsymbol{q}(t) = \boldsymbol{q}_1 + \omega\, t$ is defined as:

$$\mathbf{q}(t) = \frac{\mathbf{v} \times \boldsymbol{\omega}}{\|\boldsymbol{\omega}\|^2} + \boldsymbol{\omega} t \qquad \text{or} \qquad \text{as } \hat{\mathbf{q}}(t) = \left[\mathbf{v} \times \boldsymbol{\omega} + t\boldsymbol{\omega}\|\boldsymbol{\omega}\|^2 : \|\boldsymbol{\omega}\|^2\right] \text{ if projective notation is used.}$$

Let us consider two points in the homogeneous coordinates: $\boldsymbol{x}_1 = [x_1, y_1, z_1, w_1]^T$, $\boldsymbol{x}_2 = [x_2, y_2, z_2, w_2]^T$. The Plücker coordinates l_{ij} are defined as follows:

$\mathbf{L} = \mathbf{x}_1\mathbf{x}_2^T - \mathbf{x}_2\mathbf{x}_1^T$ $\quad$ $l_{41} = w_1x_2 - w_2x_1$ $\quad$ $l_{23} = y_1z_2 - y_2z_1$ $\quad$ $\boldsymbol{\omega} = [l_{41}, l_{42}, l_{43}]^T$

$l_{ij} = \mathbf{x}_1^{(i)}\mathbf{x}_2^{(j)} - \mathbf{x}_2^{(i)}\mathbf{x}_1^{(j)}$ $\quad$ $l_{42} = w_1y_2 - w_2y_1$ $\quad$ $l_{31} = z_1x_2 - z_2x_1$ $\quad$ $\mathbf{v} = [l_{23}, l_{31}, l_{12}]^T$

$l_{43} = w_1z_2 - w_2z_1$ $\quad$ $l_{12} = x_1y_2 - x_2y_1$

$l_{ij} = - l_{ji}$ and $l_{ii} = 0$.

Two vectors ω and $\mathbf{v}$ are defined as $\mathbf{X}_2 - \mathbf{X}_1 = \omega$, $\mathbf{X}_1 \times \mathbf{X}_2 = \mathbf{v}$,where: $\mathbf{X}_i = [x_i, y_i, z_i]^T/w_i$ are points in the Euclidean space and for a general case $w_i \neq 1$ when $\boldsymbol{x}_i$ are not ideal points, i.e. $w_i \neq 0$, we get:

$$\boldsymbol{\omega} = w_2w_1(\mathbf{X}_2 - \mathbf{X}_1) = (x_2w_1 - x_1w_2\,,\, y_2w_1 - y_1w_2\,,\, z_2w_1 - z_1w_2) = (l_{41}, l_{42}, l_{43})$$

$$\mathbf{v} = w_2w_1(\mathbf{X}_1 \times \mathbf{X}_2) = (y_1z_2 - y_2z_1\,,\, z_1x_2 - z_2x_1\,,\, x_1y_2 - y_1x_2) = (l_{23},\, l_{31}, l_{12})$$

It means that ω represents the "directional vector", while $\mathbf{v}$ represents the "positional vector". The equations above show the relation between vectors ω and $\mathbf{v}$ and the Plücker coordinates l_{ij}. In 1871 Klein [1] derived that $\omega^T \mathbf{v} = 0$, i.e. in the Plücker coordinates: $l_{23} * l_{41} + l_{31} * l_{42} + l_{12} * l_{43} = 0$. If q is a point on a line $\mathbf{q}(t) = \mathbf{q}_1 + \omega\, t$ given by the Plücker coordinates, it must satisfy the equation $\boldsymbol{\omega} \times \mathbf{q} = \mathbf{v}$. A line given by two points in homogeneous coordinates is determined as: $\mathbf{q}(t) = \frac{\mathbf{v} \times \boldsymbol{\omega}}{\|\boldsymbol{\omega}\|^2} + \boldsymbol{\omega} t$. Due to the **principle of duality** in E^3 we can exchange "point" and "plane" and therefore the same formula can be applied for an intersection computation of two planes as $\mathbf{L} = \boldsymbol{\rho}_1\boldsymbol{\rho}_2^T - \boldsymbol{\rho}_2\boldsymbol{\rho}_1^T$,. Now, the line $\boldsymbol{p}$ is given as $\mathbf{q}(t) = \frac{\mathbf{v} \times \boldsymbol{\omega}}{\|\boldsymbol{\omega}\|^2} + \boldsymbol{\omega} t$, where $\omega = [l_{41}, l_{42}, l_{43}]^T$ $\mathbf{v} = [l_{23}, l_{31}, l_{12}]^T$. Homogeneous coordinates can be used also for barycentric coordinates computation [2] and some additional hints can be found in [3].

Keywords: Projective space, homogeneous coordinates, duality principle, intersection computation.

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