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Education

Barycentric coordinates computation in homogeneous coordinates

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Abstract

Homogeneous coordinates are often used in computer graphics and computer vision applications especially for the representation of geometric transformations. The homogeneous coordinates enable us to represent translation, rotation, scaling and projection operations in a unique way and handle them properly. Today's graphics hardware based on GPU offers a very high computational power using pixel and fragment shaders not only for the processing of graphical elements, but also for the general computation using GPU as well.

It is well known that points, triangles and strips of triangles are mostly used in computer graphics processing. Generally, triangles and tetrahedra are mostly represented by vertices. Several tests like "point inside…" or "intersection of…" are very often used in applications. On the other hand, barycentric coordinates in E^2 or E^3 can be used to implement such tests, too. Nevertheless, in both cases division operations are used that potentially lead to the instability of algorithms.

The main objective of this paper is to show that if the vertices of the given polygon and/or a point itself are given in homogeneous coordinates the barycentric coordinates can be computed directly without transferring them from the homogeneous [$w \neq 1$] to the Euclidean coordinates. Instead of solving a linear system of equations, the cross-product can be used and the division operation is not needed. This is quite convenient approach for GPU computation.
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Keywords: Barycentric coordinates; Homogeneous coordinates; Projective space; Point-in-polygon; Point-in-tetrahedron; Ray-triangle intersection; Ray-tetrahedron intersection; Duality; Cross-product

1. Introduction

Geometric algorithms used in E^2 and E^3 are usually based on the Euclidean space representation. It is well known that points, triangles and strips of triangles are used in computer graphics processing. Generally, triangles and tetrahedra are mostly represented by vertices. Several tests like "point in…" or "intersection of…" are very often used in applications and barycentric coordinates in E^2 or E^3 can be used to implement such tests, too. Nevertheless, in both cases division operations are used that potentially lead to the instability of algorithms. In some cases, the positive projective space representation using homogeneous coordinates is more convenient.

The proposed approach postpones the division operation for the very last step of the computational pipeline if necessary.

2. Projective geometry and duality

Homogeneous coordinates are widely used in computer graphics applications, usually connected with geometric transformations, such as rotation, scaling, translation and projection, etc. In many cases, homogeneous coordinates are only seen as a "mathematical tool" that makes a simple description of geometric transformations possible. There are many "invisible" impacts on the algorithm design that may lead to new, faster and robust algorithms, which can also be supported in GPU hardware. Fig. 1a. presents a geometrical interpretation of Euclidean and projective spaces.

The point $\mathbf{x}$ is defined as a point in E^2 with coordinates $\mathbf{X} = (X, Y)$ or as a point with homogeneous coordinates $[x, y, w]^T$, where w usually equals 1. The point $\mathbf{x}$ is actually a "line" without the origin $[0,0,0]^T$ in the projective space P^2, and $X = x/w$ and $Y = y/w$ where $w \neq 0$. It can be seen that the line $\mathbf{p} \in E^2$ is actually a plane $\boldsymbol{\rho}$ without the origin $[0,0,0]^T$ in the projective space P^2, i.e. a line $\mathbf{p}$ in the

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Nomenclature

$\mathbf{X} = (X, Y)$ a point in Euclidean coordinates
X coordinate in Euclidean coordinates
$\mathbf{x} = [x, y, w]^T$ point in projective space represented by homogeneous coordinates
x coordinate in homogeneous coordinates
$\mathbf{p} = [a,b,c]^T$ vector of the line $\mathbf{p}$ defined as $aX+bY+c=0$
$\mathbf{a}^T \cdot \mathbf{b}$ dot product of two $\mathbf{a}$, $\mathbf{b}$ vectors
$\mathbf{a} \times \mathbf{b}$ cross-product of two $\mathbf{a}$, $\mathbf{b}$ vectors
$a_i = (-b_i\text{:}\ b_4)$, $i = 1,\ldots,3$ Plücker notation for homogeneous coordinates e.g. $a_1 = -(b_1/b_4)$

Euclidean is defined as

$$aX + bY + c = 0. \quad (1)$$

Any $w \neq 0$ can multiply this equation without any effect on the geometry and we get a representation in the projective P^2 space as follows:

$$ax + by + cw = 0, \quad w \neq 0. \quad (2)$$

In dual representation, see Fig. 1b, the plane $\boldsymbol{\rho}$ can be represented as a line $D(\boldsymbol{\rho}) \in D(P^2)$ or as a point $D(\mathbf{p}) \in D(E^2)$, when a projection is made, e.g. for $c = 1$. A complete theory on projective spaces can be found in [1–3,6].

On the other hand, there is a principle of duality that is useful when deriving a formula. The principle states that any theorem remains true when we interchange the words "point" and "line", "lie on" and "pass through", "join" and "intersection" and so on. Once the theorem has been established, the dual theorem is obtained as described above, see [5].

In other words, the principle of duality in E^2 says that in all theorems it is possible to substitute the term "point" by the term "line" and the term "line" by the term "point" and the given theorem stays valid. This helps a lot in solving some geometrical cases.

Definition 1. The cross-product of the two vectors $\mathbf{x}_1 = [x_1,y_1,w_1]^T$ and $\mathbf{x}_2 = [x_2,y_2,w_2]^T$ is defined as

$$\mathbf{x}_1 \times \mathbf{x}_2 = \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_1 & y_1 & w_1 \\ x_2 & y_2 & w_2 \end{bmatrix}, \quad (3)$$

where $\mathbf{i} = [1,0,0]^T$, $\mathbf{j} = [0,1,0]^T$ and $\mathbf{k} = [0,0,1]^T$.

Please, note that the homogeneous coordinates are used.

Theorem 1. *Let two points $\mathbf{x}_1$ and $\mathbf{x}_2$ be given in the projective space. Then the coefficients of the line $\mathbf{p}$, which is defined by those two points, are determined as the cross-product of their homogeneous coordinates*:

$$\mathbf{p} = \mathbf{x}_1 \times \mathbf{x}_2. \quad (4)$$

Proof 1. Let the line $\mathbf{p} \in E^2$ be defined in homogeneous coordinates as

$$ax + by + cw = 0. \quad (5)$$

We are actually looking for a solution to the following equations:

$$\mathbf{p}^T\mathbf{x}_1 = 0 \quad \text{and} \quad \mathbf{p}^T\mathbf{x}_2 = 0, \quad (6)$$

where $\mathbf{p} = [a,b,c]^T$.

It means that any point $\mathbf{x}$ that lies on the line $\mathbf{p}$ must satisfy both the equations above and the equation

$$\mathbf{p}^T\mathbf{x} = 0 \quad (7)$$

in other words the vector $\mathbf{p}$ is defined as

$$\mathbf{p} = \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_1 & y_1 & w_1 \\ x_2 & y_2 & w_2 \end{bmatrix}. \quad (8)$$

We can write

$$(\mathbf{x}_1 \times \mathbf{x}_2)^T\mathbf{x} = 0. \quad (9)$$

i.e.

$$\det \begin{bmatrix} x & y & w \\ x_1 & y_1 & w_1 \\ x_2 & y_2 & w_2 \end{bmatrix} = 0. \quad (10)$$

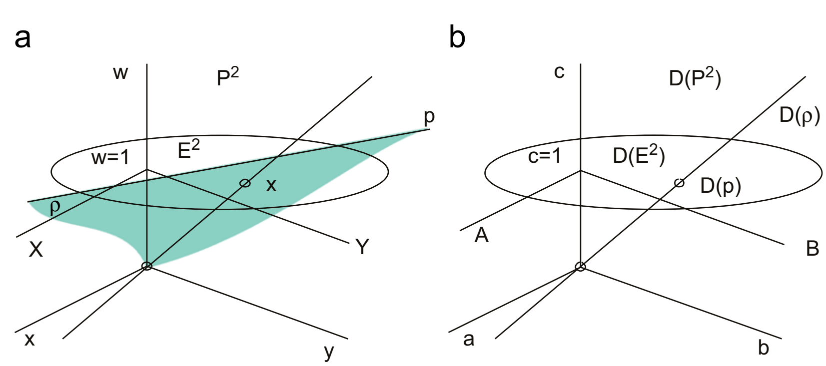

Fig. 1. Euclidean, projective and dual space representations.

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Then evaluating the determinant, we get the line coefficients as

$$a = \det\begin{bmatrix} y_1 & w_1 \\ y_2 & w_2 \end{bmatrix}, \quad b = -\det\begin{bmatrix} x_1 & w_1 \\ x_2 & w_2 \end{bmatrix},$$
$$c = \det\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix}. \qquad \square \tag{11}$$

Note: For $w = 1$ we get the standard cross-product formula and the cross-product defines the line $\mathbf{p}$, i.e.

$$\mathbf{p} = \mathbf{x}_1 \times \mathbf{x}_2, \tag{12}$$

where $\mathbf{p} = [a,b,c]^T$.

Theorem 2. *Let two lines $\mathbf{p}_1$ and $\mathbf{p}_2$ be given in the projective space. Then the homogeneous coordinates of the point $\mathbf{x}$ at the intersection of those two lines are given by the cross-product of vectors of their coordinates*

$$\mathbf{x} = \mathbf{p}_1 \times \mathbf{p}_2 = \det\begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{bmatrix}, \tag{13}$$

where $\mathbf{i} = [1,0,0]^T$, $\mathbf{j} = [0,1,0]^T$ and $\mathbf{k} = [0,0,1]^T$.

Note: Actually, two equations

$$\mathbf{p}_1^T\mathbf{x} = 0 \quad \text{and} \quad \mathbf{p}_2^T\mathbf{x} = 0 \tag{14}$$

are solved.

Proof 2. An immediate result of Theorem 1 and the duality principle.

The E^3 case is a little bit more complex as a point is dual to a plane and vice versa. It should be noted that a line in E^3 *is not* dual to a line in the dual space, for details see [5,6].

In the E^3 case, the plane $\boldsymbol{\rho}$ is given by three points $\mathbf{X} = (X, Y, Z)$ or by points in the homogeneous coordinates $\mathbf{x} = [x, y, z, w]^T$. $\square$

Theorem 3. *Let three points $\mathbf{x}_1$, $\mathbf{x}_2$ and $\mathbf{x}_3$ be given in the projective space. Then the coefficients of the plane $\boldsymbol{\rho}$, which is defined by those three points, are determined by the cross product of their homogeneous coordinates*

$$\boldsymbol{\rho} = \mathbf{x}_1 \times \mathbf{x}_2 \times \mathbf{x}_3, \tag{15}$$

where $\boldsymbol{\rho} = [a,b,c,d]^T$ and the cross-product is defined as follows:

$$\mathbf{x}_1 \times \mathbf{x}_2 \times \mathbf{x}_3 = \det\begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} & \mathbf{l} \\ x_1 & y_1 & z_1 & w_1 \\ x_2 & y_2 & z_2 & w_2 \\ x_3 & y_3 & z_3 & w_3 \end{bmatrix}, \tag{16}$$

where $\mathbf{i} = [1,0,0,0]^T$, $\mathbf{j} = [0,1,0,0]^T$, $\mathbf{k} = [0,0,1,0]^T$ and $\mathbf{l} = [0,0,0,1]^T$.

The proof is left to the reader, as it is similar to Proof 1.

Theorem 4. *Let three planes $\boldsymbol{\rho}_1$, $\boldsymbol{\rho}_2$ and $\boldsymbol{\rho}_3$ be given in the projective space. Then the homogeneous coordinates of the point $\mathbf{x}$ at the intersection of those three planes are given by the cross product of their coordinates*

$$\mathbf{x} = \boldsymbol{\rho}_1 \times \boldsymbol{\rho}_2 \times \boldsymbol{\rho}_3 \tag{17}$$

i.e.

$$\boldsymbol{\rho}_1 \times \boldsymbol{\rho}_2 \times \boldsymbol{\rho}_3 = \det\begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} & \mathbf{l} \\ a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{bmatrix}, \tag{18}$$

where $\mathbf{i} = [1,0,0,0]^T$, $\mathbf{j} = [0,1,0,0]^T$, $\mathbf{k} = [0,0,1,0]^T$ and $\mathbf{l} = [0,0,0,1]^T$.

The proof is left to the reader, as it is similar to Proof 2.

These theorems are very important as they enable us to handle some problems defined in the homogeneous coordinates efficiently and make the computations more robust and effective.

Appendix A presents a simple implementation convenient for GPU using Cg/HLSL.

3. Barycentric coordinates

In computer graphics, Euclidean or homogeneous coordinates are widely used as well as parametric formulations, e.g. triangles, parametric patches, etc. The barycentric coordinates have many useful and interesting properties, see [2,14] for details.

Let us consider a triangle with vertices $\mathbf{X}_1$, $\mathbf{X}_2$, $\mathbf{X}_3$, see Fig. 2. The position of any point $\mathbf{X} \in E^2$ can be expressed as

$$a_1X_1 + a_2X_2 + a_3X_3 = X,$$
$$a_1Y_1 + a_2Y_2 + a_3Y_3 = Y \tag{19}$$

if we add an additional condition

$$a_1 + a_2 + a_3 = 1 \tag{20}$$

we get a system of linear equations. The coefficients a_i are called barycentric coordinates of the point $\mathbf{X}$. The point $\mathbf{X}$ is inside the triangle *if and only if* $0 \leqslant a_i \leqslant 1$, $i = 1,\ldots,3$. It is useful to know that

$$a_i = \frac{P_i}{P}, \quad i = 1, \ldots, 3, \tag{21}$$

where P is the area of the given triangle and P_i is the area of the ith subtriangle.

Note: The barycentric coordinates can easily be converted into the usual parametric form. It can be seen that $a_1 = 1 - a_2 - a_3$. Substituting into Eq. (19)

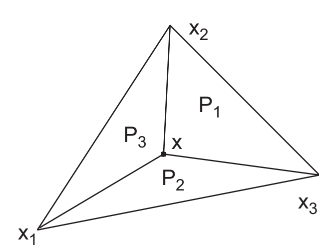

Fig. 2. Barycentric coordinates in E^2.

we obtain

$$(1 - a_2 - a_3)X_1 + a_2X_2 + a_3X_3 = X \tag{22}$$

i.e.

$$X_1 + a_2(X_2 - X_1) + a_3(X_3 - X_1) = X \tag{23}$$

and finally we get

$$a_2(X_2 - X_1) + a_3(X_3 - X_1) = X - X_1. \tag{24}$$

It is the standard formula usually used. Similarly, it may be used for other coordinates.

It can be seen that a system of linear equations, Eqs. (29)–(20), must be solved, i.e.

$$\mathbf{A}\boldsymbol{\alpha} = \boldsymbol{\beta}, \tag{25}$$

where $\boldsymbol{\alpha} = [a_1, a_2, a_3]^{\mathrm{T}}$, $\boldsymbol{\beta} = [X, Y, 1]^{\mathrm{T}}$ and

$$\mathbf{A} = \begin{bmatrix} X_1 & X_2 & X_3 \\ Y_1 & Y_2 & Y_3 \\ 1 & 1 & 1 \end{bmatrix}$$

and division operations must be used to solve this linear system of equations. In some cases, especially when the triangles are very thin, there might be a severe problem with the stability of the solution. The non-homogeneous system of linear equations $\mathbf{A}\boldsymbol{\alpha} = \boldsymbol{\beta}$ can be transformed into a homogeneous linear system

$$\begin{aligned} &b_1X_1 + b_2X_2 + b_3X_3 + b_4X = 0, \\ &b_1Y_1 + b_2Y_2 + b_3Y_3 + b_4Y = 0, \\ &b_1 + b_2 + b_3 + b_4 = 0, \end{aligned} \tag{26}$$

where $b_4 \neq 0$ and $b_i = -a_ib_4$, $i = 1,\ldots,3$.

Rewriting this system in a matrix form, we get

$$\begin{bmatrix} X_1 & X_2 & X_3 & X \\ Y_1 & Y_2 & Y_3 & Y \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} = \mathbf{0} \tag{27}$$

or in the matrix form

$$\mathbf{Bb} = \mathbf{0} \quad \text{or} \quad [\mathbf{A}|\mathbf{X}][\mathbf{b}] = \mathbf{0}, \tag{28}$$

where $\mathbf{b} = [b_1, b_2, b_3, b_4]^{\mathrm{T}}$ and $\mathbf{X} = [X, Y, 1]^{\mathrm{T}}$:

$$\mathbf{A} = \begin{bmatrix} X_1 & X_2 & X_3 \\ Y_1 & Y_2 & Y_3 \\ 1 & 1 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = [\mathbf{A}|\mathbf{X}].$$

In another way, we are looking for a vector $\boldsymbol{\tau}$, see Eq. (14), that satisfies the condition

$$\boldsymbol{\tau}^{\mathrm{T}}\mathbf{b} = 0, \tag{29}$$

where $\boldsymbol{\tau} = [\tau_1, \tau_2, \tau_3, \tau_4]^{\mathrm{T}}$.

This equation can be expressed using the determinant form as

$$\det \begin{vmatrix} \tau_1 & \tau_2 & \tau_3 & \tau_4 \\ X_1 & X_2 & X_3 & X \\ Y_1 & Y_2 & Y_3 & Y \\ 1 & 1 & 1 & 1 \end{vmatrix} = 0. \tag{30}$$

It is obvious that it can be formally written as

$$\mathbf{b} = \boldsymbol{\xi} \times \boldsymbol{\eta} \times \mathbf{w}, \tag{31}$$

where $\mathbf{b} = [b_1, b_2, b_3, b_4]^{\mathrm{T}}$, $\boldsymbol{\xi} = [X_1, X_2, X_3, X]^{\mathrm{T}}$, $\boldsymbol{\eta} = [Y_1, Y_2, Y_3, Y]^{\mathrm{T}}$ and $\mathbf{w} = [1, 1, 1, 1]^{\mathrm{T}}$ and the barycentric coordinates of the point $\mathbf{X}$ are given as

$$a_1 = -\frac{b_1}{b_4}, \quad a_2 = -\frac{b_2}{b_4}, \quad a_3 = -\frac{b_3}{b_4}.$$

We can use the Plücker coordinates notation and write

$$a_i = (-b_i : b_4), \quad i = 1, \ldots, 3.$$

If $b_4 = 0$, the triangle is degenerated to a line segment or to a point, i.e. it is a singular case, which can be correctly detected.

The given point $\mathbf{X}$ is inside the given *triangle if and only if* $0 \leqslant a_i \leqslant 1$, $i = 1,\ldots,3$. This condition is a little bit more complicated for the homogeneous representation and can be expressed by a sequence

$$\begin{aligned} &\text{if } b_4 > 0 \quad \text{then } 0 \leqslant -b_i \leqslant b_4 \\ &\qquad\qquad \text{else } b_4 \leqslant -b_i \leqslant 0, \quad i = 1, \ldots, 3. \end{aligned}$$

This is a very important result as it means that *we do not need the division operation* for testing whether the given point $\mathbf{X}$ is inside the given triangle!

In many applications, the vertices of the given triangle and the given point $\mathbf{X}$ can be given in homogeneous coordinates. Let us explore how the barycentric coordinates could be computed in this case.

The linear system of equations for the barycentric coordinates can be rewritten as

$$\begin{aligned} &a_1\frac{x_1}{w_1} + a_2\frac{x_2}{w_2} + a_3\frac{x_3}{w_3} = \frac{x}{w}, \\ &a_1\frac{y_1}{w_1} + a_2\frac{y_2}{w_2} + a_3\frac{y_3}{w_3} = \frac{y}{w}, \\ &a_1 + a_2 + a_3 = 1, \end{aligned} \tag{32}$$

where $\mathbf{x}_i = [x_i, y_i, w_i]^{\mathrm{T}}$ represents the ith vertex triangle in the homogeneous coordinates and $\mathbf{x} = [x, y, w]^{\mathrm{T}}$ is the given point in the homogeneous coordinates.

We can multiply the linear system by $w \neq 0$, $w_i \neq 0$, $i = 1,\ldots,3$ and substitute

$$\begin{aligned} &b_1 = -a_1w_2w_3w, \quad b_2 = -a_2w_1w_3w, \\ &b_3 = -a_3w_1w_2w, \quad b_4 = w_1w_2w_3w. \end{aligned} \tag{33}$$

Thus, we get

$$\begin{aligned} &b_1x_1 + b_2x_2 + b_3x_3 + b_4x = 0,\\ &b_1y_1 + b_2y_2 + b_3y_3 + b_4y = 0,\\ &b_1w_1 + b_2w_2 + b_3w_3 + b_4w = 0 \end{aligned} \tag{34}$$

and in the matrix notation

$$\begin{bmatrix} x_1 & x_2 & x_3 & x \\ y_1 & y_2 & y_3 & y \\ w_1 & w_2 & w_3 & w \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} = \mathbf{0}. \tag{35}$$

We are looking for a vector $\boldsymbol{\tau}$ that satisfies the following equation:

$$\boldsymbol{\tau}^{\mathrm{T}}\mathbf{b} = 0, \tag{36}$$

where the vector $\boldsymbol{\tau}$ is defined as$\boldsymbol{\tau} = [\tau_1, \tau_2, \tau_3, \tau_4]^{\mathrm{T}}$.

Then the solution is defined as

$$\det \begin{vmatrix} \tau_1 & \tau_2 & \tau_3 & \tau_4 \\ x_1 & x_2 & x_3 & x \\ y_1 & y_2 & y_3 & y \\ w_1 & w_2 & w_3 & w \end{vmatrix} = 0 \tag{37}$$

and we can formally write

$$\mathbf{b} = \boldsymbol{\xi} \times \boldsymbol{\eta} \times \mathbf{w}, \tag{38}$$

where $\mathbf{b} = [b_1, b_2, b_3, b_4]^{\mathrm{T}}$, $\boldsymbol{\xi} = [x_1, x_2, x_3, x]^{\mathrm{T}}$, $\boldsymbol{\eta} = [y_1, y_2, y_3, y]^{\mathrm{T}}$ and $\mathbf{w} = [w_1, w_2, w_3, w]^{\mathrm{T}}$.

Of course, the conditions in the case that the point is inside the given triangle are slightly more complex, and the condition $0 \leqslant a_i \leqslant 1$ $i = 1,\ldots,3$ can be expressed by the following criteria:

$$\begin{aligned} &0 \leqslant (-b_1 : w_2w_3w) \leqslant 1,\\ &0 \leqslant (-b_2 : w_1w_3w) \leqslant 1,\\ &0 \leqslant (-b_3 : w_1w_2w) \leqslant 1. \end{aligned} \tag{39}$$

This means that the barycentric coordinates can be computed *without using the division operation* even if the vertices of the given triangle and the point $\mathbf{x}$ are given in homogeneous coordinates. Therefore, the approach presented here is more robust than the direct computation, i.e. normalizing the vertices and point coordinates into Euclidean coordinates and standard barycentric coordinates computation. In addition, the test if a point is inside the given triangle is consequently more robust.

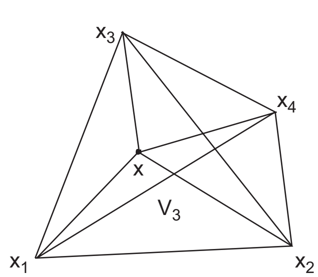

Fig. 3. Barycentric coordinates in E^3.

Of course, there is a natural question: is it possible to extend the above-mentioned approach to the E^3 case?

Let us consider the E^3 case, where the ‘‘point in a tetrahedron’’ test is similar to the ‘‘point in a triangle’’ test in E^2, see Fig. 3.

It can be seen that the barycentric coordinates are given as

$$\begin{aligned} &a_1X_1 + a_2X_2 + a_3X_3 + a_4X_4 = X,\\ &a_1Y_1 + a_2Y_2 + a_3Y_3 + a_4Y_4 = Y,\\ &a_1Z_1 + a_2Z_2 + a_3Z_3 + a_4Z_4 = Z,\\ &a_1 + a_2 + a_3 + a_4 = 1. \end{aligned} \tag{40}$$

It is useful to know that

$$a_i = \frac{V_i}{V}, \quad i = 1, \ldots, 3, \tag{41}$$

where V is the volume of the given tetrahedron and V_i the volume of the ith sub-tetrahedron.

The non-homogeneous system of linear equations can be transformed into a homogeneous linear system of equations

$$\begin{aligned} &b_1X_1 + b_2X_2 + b_3X_3 + b_4X_4 + b_5X = 0,\\ &b_1Y_1 + b_2Y_2 + b_3Y_3 + b_4Y_4 + b_5Y = 0,\\ &b_1Z_1 + b_2Z_2 + b_3Z_3 + b_4Z_4 + b_5Z = 0,\\ &b_1 + b_2 + b_3 + b_4 + b_5 = 0, \end{aligned} \tag{42}$$

where $b_5 \neq 0$ and $b_i = -a_ib_5$, $i = 1,\ldots,4$.

Rewriting this system in matrix form, we get

$$\begin{bmatrix} X_1 & X_2 & X_3 & X_4 & X \\ Y_1 & Y_2 & Y_3 & Y_4 & Y \\ Z_1 & Z_2 & Z_3 & Z_4 & Z \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{bmatrix} = \mathbf{0} \tag{43}$$

i.e.

$$\mathbf{Bb} = \mathbf{0} \quad \text{or} \quad [\mathbf{A}|\mathbf{X}][\mathbf{b}] = \mathbf{0}, \tag{44}$$

where $\mathbf{b} = [b_1, b_2, b_3, b_4, b_5]^{\mathrm{T}}$ and $\mathbf{X} = [X, Y, Z, 1]^{\mathrm{T}}$,

$$\mathbf{A} = \begin{bmatrix} X_1 & X_2 & X_3 & X_4 \\ Y_1 & Y_2 & Y_3 & Y_4 \\ Z_1 & Z_2 & Z_3 & Z_4 \\ 1 & 1 & 1 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = [\mathbf{A}|\mathbf{X}].$$

Again, we are looking for a vector $\boldsymbol{\tau}$ that satisfies the equation

$$\boldsymbol{\tau}^{\mathrm{T}}\mathbf{b} = 0, \tag{45}$$

where $\boldsymbol{\tau} = [\tau_1, \tau_2, \tau_3, \tau_4, \tau_5]^{\mathrm{T}}$.

The equation can be expressed using a determinant form as

$$\det\begin{vmatrix} \tau_1 & \tau_2 & \tau_3 & \tau_4 & \tau_5 \\ X_1 & X_2 & X_3 & X_4 & X \\ Y_1 & Y_2 & Y_3 & Y_4 & Y \\ Z_1 & Z_2 & Z_3 & Z_4 & Z \\ 1 & 1 & 1 & 1 & 1 \end{vmatrix} = 0. \tag{46}$$

It can be seen that we can formally write again

$$\mathbf{b} = \boldsymbol{\xi} \times \boldsymbol{\eta} \times \boldsymbol{\zeta} \times \mathbf{w}, \tag{47}$$

where $\mathbf{b} = [b_1, b_2, b_3, b_4, b_5]^T$, $\boldsymbol{\xi} = [X_1, X_2, X_3, X_4, X]^T$, $\boldsymbol{\eta} = [Y_1, Y_2, Y_3, Y_4, Y]^T$, $\boldsymbol{\zeta} = [Z_1, Z_2, Z_3, Z_4, Z]^T$ and $\mathbf{w} = [1, 1, 1, 1, 1]^T$.

This means that the barycentric coordinates of the point $\mathbf{X}$ are given as

$$a_1 = -\frac{b_1}{b_5}, \quad a_2 = -\frac{b_2}{b_5}, \quad a_3 = -\frac{b_3}{b_5}, \quad a_4 = -\frac{b_4}{b_5} \tag{48}$$

or if we use the Plücker coordinates notation, they are given as $a_i = (-b_i : b_5)$, $i = 1, \ldots, 4$.

The given point $\mathbf{X}$ is inside the given tetrahedron *if and only if* $0 \leqslant a_i \leqslant 1$, i = 1,…,4.

This condition can be expressed by the following sequence:

if $b_5 > 0$ then $0 \leqslant -b_i \leqslant b_5$
else $b_5 \leqslant -b_i \leqslant 0$.

If $b_5 = 0$, the tetrahedron is degenerated to a triangle or to a line segment or to a point, i.e. singular cases that can be correctly detected.

Let us again consider a case when the tetrahedron vertices and the given point are in homogeneous coordinates.

The linear system of equations can be rewritten as

$$\begin{aligned} a_1\frac{x_1}{w_1} + a_2\frac{x_2}{w_2} + a_3\frac{x_3}{w_3} + a_4\frac{x_4}{w_4} &= \frac{x}{w}, \\ a_1\frac{y_1}{w_1} + a_2\frac{y_2}{w_2} + a_3\frac{y_3}{w_3} + a_4\frac{y_4}{w_4} &= \frac{y}{w}, \\ a_1\frac{z_1}{w_1} + a_2\frac{z_2}{w_2} + a_3\frac{z_3}{w_3} + a_4\frac{z_4}{w_4} &= \frac{z}{w}, \\ a_1 + a_2 + a_3 + a_4 &= 1, \end{aligned} \tag{49}$$

where $\mathbf{x}_i = [x_i, y_i, z_i, w_i]^T$ represents the ith vertex coordinates in the homogeneous coordinates.

We can multiply the linear system of equations by $w \neq 0$, $w_i \neq 0$ $i = 1, \ldots, 4$ and substitute

$$\begin{aligned} b_1 &= -a_1 w_2 w_3 w_4 w, \quad b_2 = -a_2 w_1 w_3 w_4 w, \\ b_3 &= -a_3 w_1 w_2 w_4 w, \quad b_4 = -a_4 w_1 w_2 w_3 w, \\ b_5 &= w_1 w_2 w_3 w_4. \end{aligned} \tag{50}$$

This results into a standard homogeneous linear system:

$$\begin{aligned} b_1 x_1 + b_2 x_2 + b_3 x_3 + b_4 x_4 + b_5 x &= 0, \\ b_1 y_1 + b_2 y_2 + b_3 y_3 + b_4 y_4 + b_5 y &= 0, \\ b_1 z_1 + b_2 z_2 + b_3 z_3 + b_4 z_4 + b_5 z &= 0, \\ w_1 b_1 + w_2 b_2 + w_3 b_3 + w_4 b_4 + w b_5 &= 0 \end{aligned} \tag{51}$$

that can be expressed in the matrix form as

$$\begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x \\ y_1 & y_2 & y_3 & y_4 & y \\ z_1 & z_2 & z_3 & z_4 & z \\ w_1 & w_2 & w_3 & w_4 & w \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{bmatrix} = \mathbf{0}. \tag{52}$$

We are looking for a vector $\boldsymbol{\tau}$ that satisfies the equation

$$\boldsymbol{\tau}^T \mathbf{b} = 0, \tag{53}$$

where the vector $\boldsymbol{\tau} = [\tau_1, \tau_2, \tau_3, \tau_4, \tau_5]^T$ is defined as

$$\det\begin{vmatrix} \tau_1 & \tau_2 & \tau_3 & \tau_4 & \tau_5 \\ x_1 & x_2 & x_3 & x_4 & x \\ y_1 & y_2 & y_3 & y_4 & y \\ z_1 & z_2 & z_3 & z_4 & z \\ w_1 & w_2 & w_3 & w_4 & w \end{vmatrix} = 0. \tag{54}$$

It can be seen that we can formally write

$$\mathbf{b} = \boldsymbol{\xi} \times \boldsymbol{\eta} \times \boldsymbol{\zeta} \times \mathbf{w}, \tag{55}$$

where $\mathbf{b} = [b_1, b_2, b_3, b_4, b_5]^T \boldsymbol{\xi} = [x_1, x_2, x_3, x_4, x]^T \boldsymbol{\eta} = [y_1, y_2, y_3, y_4, y]^T \boldsymbol{\zeta} = [z_1, z_2, z_3, z_4, z]^T \mathbf{w} = [w_1, w_2, w_3, w_4, w]^T$.

The conditions—if the point is inside the given triangle—are slightly more complex and the condition $0 \leqslant a_i \leqslant 1$, $i = 1, \ldots, 4$ can be expressed by the following criteria:

$$\begin{aligned} &0 \leqslant (-b_1 : w_2 w_3 w_4 w) \leqslant 1, \\ &0 \leqslant (-b_2 : w_1 w_3 w_4 w) \leqslant 1, \\ &0 \leqslant (-b_3 : w_1 w_2 w_4 w) \leqslant 1, \\ &0 \leqslant (-b_4 : w_1 w_2 w_3 w) \leqslant 1. \end{aligned} \tag{56}$$

It is worth noting that the equations for the computation of barycentric coordinates given above can be simplified for special cases, e.g. if the tetrahedron vertices are expressed in the Euclidean coordinates or the given point $\mathbf{x}$ is expressed in the Euclidean coordinates. Such simplifications will increase the speed of computation significantly without compromising the robustness of the computation. Nevertheless, the resulting barycentric coordinates are generally in the projective space, i.e. the homogeneous coordinate is not equal to "1" in general.

4. Conclusion

This paper describes a robust barycentric coordinates computation for a triangle and for a tetrahedron in the Euclidean and projective spaces. The presented approach can be applied not only to the tests "point inside…" but also for the line/ray intersection problems as well.

The main advantages of the proposed approach are:

- The barycentric coordinates can be computed *without a division operation*; the use of a division operation is postponed for the final evaluation step if necessary.
- If the point for which barycentric coordinates are computed or the vertices of a triangle or a tetrahedron is given in the homogeneous coordinates, no division by a homogeneous coordinate is needed and the computation is done directly using homogeneous coordinates.
- The computation of barycentric coordinates is more robust especially for thin triangles or tetrahedra as we do not use a division operation that causes instability and decreases the robustness of the computation in general.
- There is some hope that this approach could help to solve certain problems with the robustness and instability of some algorithms in specific cases, too.
- The presented approach can also be applied to an effective use of GPU as instead of solving a linear system of equations a cross product operation can be used, see [9,10].
- The principle of duality and the use of homogeneous coordinates can lead to new directions in the design of algorithms, leading to simple, robust and faster algorithms, e.g. the line clipping algorithm in E^2 [7,8], polygon or polyhedra intersection tests [4,11–13].

Appendix A presents a simple sequence of the cross product computation in Cg/HLSL.

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Appendix A

The cross product in 4D defined as

$$\mathbf{x}_1 \times \mathbf{x}_2 \times \mathbf{x}_3 = \det \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} & \mathbf{l} \\ x_1 & y_1 & z_1 & w_1 \\ x_2 & y_2 & z_2 & w_2 \\ x_3 & y_3 & z_3 & w_3 \end{vmatrix} \tag{A.1}$$

can be implemented in Cg/HLSL on GPU as follows:

```
float4 cross_4D(float4 x1, float4 x2, float4 x3)
{
        float4 a;
        a.x = dot(x1.yzw, cross(x2.yzw, x3.yzw));
        a.y = -dot(x1.xzw, cross(x2.xzw, x3.xzw));
        // or a.y = dot(x1.xzw, cross(x3.xzw, x2.xzw));
        a.z = dot(x1.xyw, cross(x2.xyw, x3.xyw));
        a.w = -dot(x1.xyz, cross(x2.xyz, x3.xyz));
        // or a.w = dot(x1.xyz, cross(x3.xyz, x2.xyz));
        return a;
}
```

or more compactly

```
float4 cross_4D(float4 x1, float4 x2, float4 x3)
{
        return ( dot(x1.yzw, cross(x2.yzw, x3.yzw)),
        -dot(x1.xzw, cross(x2.xzw, x3.xzw)),
        dot(x1.xyw, cross(x2.xyw, x3.xyw)),
        -dot(x1.xyz, cross(x2.xyz, x3.xyz)) );
}
```

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