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A Topological Description of 2D Manifold Objects

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Abstract

During the last years 3D data have become attractive in computer graphics. This trend leads to the emergence of many systems that can organize, recognize, retrieve, classify or encrypt these data. The core of all these systems is a method that can describe a 3D object by the limited amount of information. This paper is dedicated to a proposition of a method that describes 3D objects on the basis of their topology and geometry. The Moorse theory, which is a bridge between topological and geometrical description, is used here. A curvature of a surface describes geometrical properties of the object and the Reeb graph describes its topology. A combination of both descriptions together gives a perspective tool for effective handling of large geometrical data sets.

1. Introduction

3D models can be represented many ways (triangular meshes, volumetric data, implicit functions, etc.) and we can meet with them in many branches, such as medicine, industry, or military. They fill out a gap where image information is not sufficient. On the other hand images are used longer time then 3D data in computer graphics and thus many problems still exist in which a stable and robust algorithm for 3D data is not developed yet. In this case we still have to work with images or to use methods that are still in development.
A feature extraction belongs to those problems. It tries to describe 3D models by limited the number of information that characterizes them. This issue is used in many applications, e.g., recognize, retrieval, encrypt, or classification systems. At present several feature extraction methods, which work with 3D data, already exist. They are based on different mathematical backgrounds and each method of course has any advantages and any drawbacks (see [5] for details).
This paper presents a proposal of a new method that can be used to the description of 3D objects. Our method combines advantages of methods that are based on a topological and histogram description.
The remainder of the paper can be divided into three parts. The first part of the paper introduces the Moorse theory and a construction of the Reeb graph. The second one is dedicated to a curvature of an object surface. The last part of the paper connects all together to obtain a description of 3D objects, which is suitable for the feature extraction. This part also includes a short description of our future work.

2. Morse theory

In the 1920s, M. Morse developed the Morse theory. This theory allows to find key points on smooth manifolds and to determine a type of these points. It is based on the Taylor series. Short introduction into this theory is following [8], [7]:
Let us have a smooth function f of a single variable x. This function can be described near a given point x_0 by the following Taylor series:

$$f(x) = \sum_{k=0}^{\infty} c_k(x_0)(x-x_0)^k$$

where:

$$c_k(x_0) = \frac{f^{(k)}(x_0)}{k!}$$

According to values of constants c_k, the point x_0 can be classified into following groups:
The point x_0 is a *regular point* if $c_0(x_0) = 0$ and $c_1(x_0)$ does not vanish:

$$f(x) \cong c_1(x_0)(x-x_0).$$

The point x_0 is a *non-degenerate critical point* if $c_0(x_0) = c_1(x_0) = 0$ and $c_2(x_0)$ does not vanish:

$$f(x) \cong c_2(x_0)(x - x_0)^2.$$

The point x_0 is a *degenerate critical points* if $c_0(x_0) = c_1(x_0) = c_2(x_0) = 0$ and $c_3(x_0)$ does not vanish:

$$f(x) \cong c_3(x_0)(x - x_0)^3.$$

Note that sometimes critical and non-critical points are also called *singularities* of the function *f(x)*. A smooth function *f(x)* is called the *Morse function* iff all singularities are non-degenerate critical points.

Now we can think more generally. Let us suppose that x is not a single variable but it is a point in E^N. According to analogy, the point $\mathbf{x}$ is non-degenerate if the gradient of the function vanish at this point, i.e. $grad(f(\mathbf{x})) = \mathbf{0}$, and the determinant of the Hessian is not zero, i.e. $det(\mathrm{H}(f(\mathbf{x}))) \neq 0$.

Let us denote the number of negative eigenvalues of Hessian $\mathrm{H}(f(\mathbf{x}))$ as the index l. Then the following lemma can be introduced [8], [7]:

Morse Lemma: Let $\mathbf{p} \in E^N$ be a non-degenerate critical point of a Morse function $f(\mathbf{x})$ with the index l and let $c = f(\mathbf{p})$. Then there is a local coordinate system $\mathbf{y} = (y_1, y_2, \ldots, y_N)$ in a neighbourhood U of $\mathbf{p}$ as its origin and:

$$f(\mathbf{y}) = c - y_1^2 - y_2^2 - \cdots - y_l^2 + y_{l+1}^2 + \cdots + y_N^2$$

A practical meaning of this theory can be shown on a 2D differentiable manifold ($N = 2$) where the Morse function $f(\mathbf{x})$ represents the height from an xz plane. A surface shape of neighbourhood U of point $\mathbf{p}$ can be estimated on the basis of the index l (see the figure 1). Note that in this case the Hessian matrix has the rank 2 with the following values:

$$\mathbf{H}(f(\mathbf{x})) = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial z} \\ \frac{\partial^2 f}{\partial x \partial z} & \frac{\partial^2 f}{\partial z^2} \end{bmatrix}.$$

According to eigenvalues of the matrix H we can classify type of the non-degenerate critical point as a pit, a saddle, or a peak. It is also demonstrated in the figure 1 (for more details see e.g. [11]).

3. Reeb graph

The Morse theory is a fundamental for the Reeb graph. In the 1940s, G. Reeb proposed a graph that encrypts information about behaviour of the Morse function on a manifold [9]. Its definition can be following (taken from [2]):

Reeb graph: Let M be a compact manifold and let $f: M \rightarrow R$ be a real Morse function. The Reeb graph of the function f is a quotient space of the graph of f in $M x R$ by the equivalence relation "~" given by:

$$(\mathbf{x}_1, f(\mathbf{x}_1)) \sim (\mathbf{x}_2, f(\mathbf{x}_2))$$

iff $f(\mathbf{x}_1) = f(\mathbf{x}_2)$ and $\mathbf{x}_1$ and $\mathbf{x}_2$ are in the same connected component of $f^{-1}(f(\mathbf{x}_1))$.

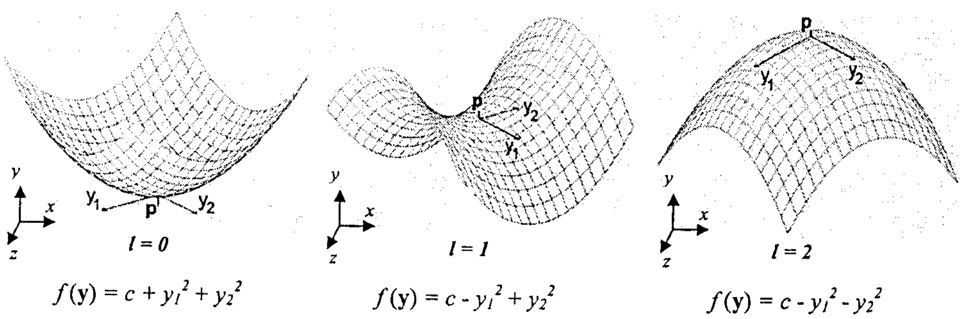

Figure 1: Examples of surface shapes of neighborhood U for different values of eigenvalues $\mathrm{H}(f(\mathbf{x}))$, the Morse function $f(\mathbf{x})$ is represented as a height from a xz plane.

Briefly, we can describe the Reeb graph as follows. Two points on the manifold M are represented by the same element in the graph, if and only if they have the same value of the function f and they belong to the same component of the inverse image of the function $f(\mathbf{x}_1)$. Note that the inverse image represents contour lines on the manifold in which the value of the function f is constant. An example of the Reeb graph is shown in the figure 2, where a smooth object and the adequate Reeb graph are sketched. The Morse function is represented as a height function measured as a distance from the xz plane (contour lines representing $f(\mathbf{x}) = const.$ are outlined by dot lines in the object).

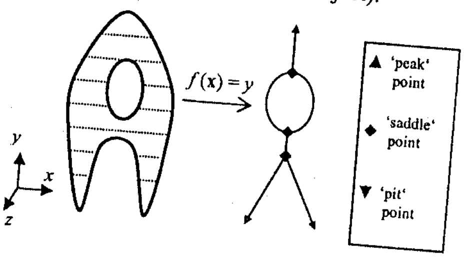

Figure 2: An example of the Reeb graph, where the Morse function is defined as $f(\mathbf{x}) = y$.

Note that the height function is selected due to a better illustration. If we think that an object is in 3D space, the Reeb graph is a projection from 4D space (MxR) into a graph representation according to the mentioned definition. This projection is generally hard to draw. Therefore, a height function, which everyone can imagine very well, is selected. Of course, we can select a totally different function. Then we have to note that the Reeb graph can have a different structure. This fact can be demonstrated very easy. If we returned back to the figure 2 and we selected the Morse function $f(\mathbf{x}) = x$, the Reeb graph would be changed as the figure 3 shows.

This fact leads us to think about any required properties on the Morse function. However, we will discuss about them later. Let us describe a construction of the Reeb graph.

Morse theory can classify points on the surface of a manifold into several groups. Non-degenerate critical points are the most important for us. When we look at the Reeb graph, we can see that nodes of the graph are just in these points.

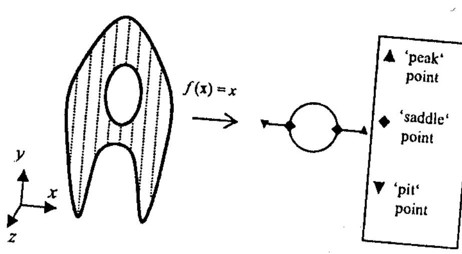

Figure 3: An example of the Reeb graph, where the Morse function is defined as $f(\mathbf{x}) = x$.

Let us define an orientation of the edges in the graph according to values of the Morse function in the nodes – e.g., edge always starts from a node which has a smaller value than the second one (or vice versa). Then each leaf of the Reeb graph (i.e., a node from which starts or in which ends the only one edge) represents a point that is a peak of a pit of the function. Two situations can come in a point where a saddle of the fucntion is either an edge of the graph is bifurcated on two edges (one edge ends in this node and two edges start from this node), or two edges are connected together and continue as one edge (two edges end in this node and one edge starts from this node).

According to this observation we would not have any problems to construct the Reeb graph. However, we still work with analytic mathematics. In a practice, objects are often approximated by triangular meshes and the others problems have to be solved. There are algorithms that construct the Reeb graphs (e.g. [6]) and its time complexity is $O(N \log(N))$, where N represents the number of the edges of the triangular mesh.

Note that there are also many variations of the Reeb graph. They are usually known as the Extended Reeb Graph (shortly ERG) and they add another information to the Reeb graph or modify the basic algorithm by needs.

4. Curvature of surfaces

Smooth manifolds have a great advantage. Neighbourhood of each point on the surface is continuous into a required derivation.

This property permits us to use knowledge from differential mathematics. A short introduction to the differential geometry is also included here. Then this information is used to a proposal of a topological description that can be used for the feature extraction.

An arbitrary curve in the 3D space can be described unambiguously by so called the first and the second curvature. The main advantage of this description is invariance to rotation and position of the curve. If we moved or rotated a curve we still would get the same description.

Unfortunately, an invariant description of a surface based on curvatures is not so easy (details in related publication, e.g. [1], [3], [10]). Some interesting variables only will be mention here, such as the Mean curvature H and Gaussian curvature K that can be used to solve our problem. These curvatures can be defined many ways. One of them is for example by the following equations:

$$K = k_1 \cdot k_2,$$

$$H = \frac{1}{2}(k_1 + k_2),$$

where k_1 and k_2 are principal curves of normal section in the point **x** (see mentioned books for details). According to values of these curvatures the points of a surface can be classified into several groups as the following table shows.

	$K<0$	$K=0$	$K>0$
$H<0$	saddle ridge	ridge	peak
$H=0$	min. surface	flat	none
$H>0$	saddle valley	valley	pit

This table also can serve as a help to analyze a local neighbourhood of a point on the surface of an object. Graphical shapes of individual type of the point are shown in the figure 4 (their importance for us is discussed later).

Note that the curvatures can be calculated either numerical or analytical ways. Due to triangular meshes that usually represent a surface of 3D objects, numerical methods are preferred. However, there are many algorithms that estimate curvatures, e.g. [12] presents an overview and a comparison among them.

5. Topological description

In the introduction we mentioned that feature extraction methods were used in many applications and the main task of these methods was to describe objects by limited number of information. Note that a proposal of that method also has to be connected with its employment. A problem of similarity comparing among objects usually has to be solved and this still has to be kept in the mind.

The first part of this paper introduced the Reeb graph. We can see this graph as a 'skeleton'

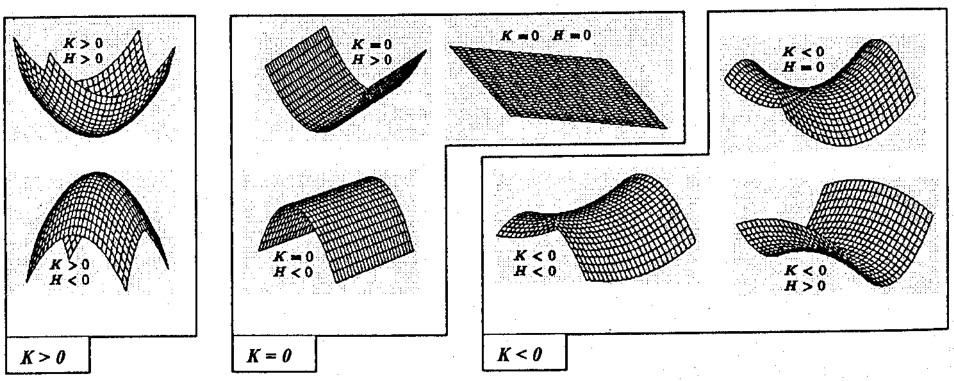

Figure 4: A classification of points on a surface of a smooth manifold into several groups according to values of the Gaussian curvature K and the Mean curvature M (the figures are taken from [1]).

of an object that tracks an object topology. However, it also was shown that the height function is not convenient for the feature extraction. It does not guarantee invariance to a rotation of the object. Therefore, the second part of this paper was dedicated to the differential geometry, exactly, to the Mean and Gaussian curvature that are already invariant to geometrical transformations. Now we will connect both theories together to get a suitable description for the feature extraction.

At first we have to define the Morse function. When we look at the figure 4 again we can divide all types of the points into three basic groups (as is outlined in the figure). First group consists of peaks and pits. These points represent an outward or inward bulging of the surface. The second one is represented by planes or surfaces that are curved only in one direction (i.e., a valley, or a ridge). The last group consists of the remainder basic surfaces that are curved in one direction on one side and in another direction on the opposite site (i.e., a saddle). An example of the Reeb graph where the Morse function is defined by the mentioned curvature is shown in the figure 5. Note that a loop in the Reeb graph does not represent the gap in the torus (like in the height function – you can try to draw a figure). In this case the top node of the graph represents all the furthest points on the surface from the centre of the torus. The bottom node represents all the nearest points on the surface to the centre of the torus.

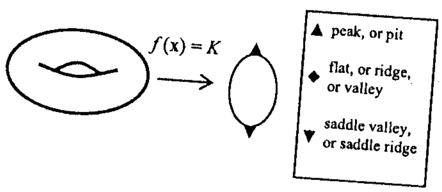

Figure 5: The Reeb graph of a torus, where the Morse function is the Gaussian curvature K of the surface.

It can be seen that the Gaussian curvature was only used as the Morse function. It does not mean that the Mean curvature is not important. The Gaussian curvature determines a type of nodes and the Mean curvature determines a direction of a surface curvature. Therefore, this value is also important and it should be saved in each node of the graph due to a later comparison of graphs mutually.

An advantage of our description from the point of view as object similarity can be seen in the figure 6. Small deformations of a sphere are registered in changes of the Reeb graph and similarity can be assessed according to the evaluation (by the Mean curvature) and the type of nodes in the graph and according to the structure of the graph.

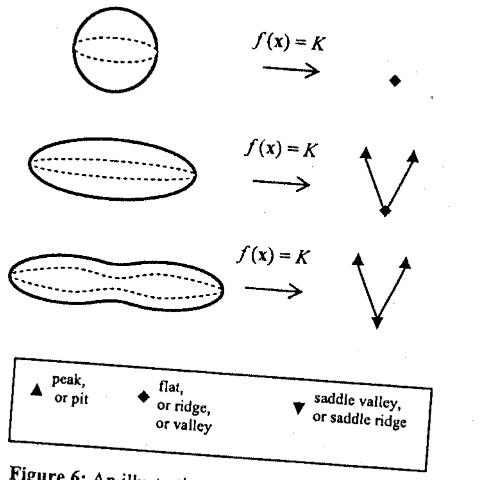

Figure 6: An illustration of the Reeb graphs for several objects (made by a simple deformation of a sphere), the Morse function is the Gaussian curvature K of the surface.

6. Conclusion and future work

A new topological description of 3D objects is introduced in this paper. Its main advantage is invariance to geometrical transformations, i.e. translation, rotation, or scaling. The construction of this description is fast and time complexity of the algorithm is $O(N^2)$, where N is the number of points representing an object. Unfortunately, there are also some drawbacks. Especially, a problem is that we only can use this description for smooth manifolds. The second derived function is needed for calculation of the Gaussian and Mean curvatures. In spite of this we believe that this method or its modification can be used in many applications.

In the future, we will use this description for a similarity comparison problem. Exactly, we want to propose a feature extraction method that could be used in a retrieval system for 3D models represented by triangular meshes.

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